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ARITHMETIC

IN

THEORY AND PRACTICE

LONDON: PRINTED BY
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AND PARLIAMENT STREET

A TREATISE

ON

ARITHMETIC

IN THEORY AND PRACTICE

WITH AN APPENDIX

CONTAINING AN INTRODUCTION TO MENSURATION

BY

JAMES THOMSON, LL.D.

LATE PROFESSOR OF MATHEMATICS IN THE UNIVERSITY OF GLASGOW

SEVENTY-SECOND EDITION

EDITED BY HIS SONS

JAMES THOMSON, LL.D., F.R.S.

PROFESSOR OF ENGINEERING IN THE UNIVERSITY OF GLASGOW

AND

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PROFESSOR OF NATURAL PHILOSOPHY IN THE UNIVERSITY OF GLASGOW

LONDON

LONGMANS, GREEN, AND CO.

1880

PREFACE

TO

THE SEVENTY-SECOND EDITION.

THIS Edition has been edited by Professor James Thomson, F.R.S., Professor of Civil Engineering in the University of Glasgow, and Professor Sir William Thomson, F.R.S., Professor of Natural Philosophy in the University of Glasgow and Fellow of St. Peter's College, Cambridge, two sons of the Author. They have taken greater liberty with the original text than other editors would have been entitled to take. Considerable changes and additional explanations have been introduced by Professor James Thomson in the earlier chapters of the work, up to Discount. Several of the later and less important chapters have been abridged. In the chapter on Annuities new information has been introduced, to bring it up to date, according to the practical experience of modern life offices. The chapter on Exchanges, which had been revised many years ago, after the gold discoveries, by Dr. W. Neilson Hancock, formerly Professor of Political Economy in the University of Dublin, has been again revised by him for changes consequent on the depreciation of silver. The whole work has been edited by Professor Sir William Thomson.

UNIVERSITY, GLASGOW: 1880.

AUTHOR'S PREFACE

TO

THE TWENTY-THIRD EDITION.

THE object of the following treatise is to present a full and regular course of whatever is useful in arithmetic. With this view, rules are given for performing all the requisite operations, and are illustrated by many examples; and numerous exercises are prescribed, to afford the pupil that practice which alone can produce expertness and accuracy in the management of numbers.

The more important of the definitions and rules are in the largest type employed in the work, and are such as some teachers may perhaps consider it proper for the learner to commit to memory: the examples and exercises, and the principal illustrations, are in a character somewhat smaller: the less important illustrations and remarks are in a type still smaller, and may perhaps be omitted by the younger pupil: and the notes contain miscellaneous information, which may be interesting to many readers.

The reasons of the rules and operation are explained, not in strict, formal demonstrations, but generally by simple and easy illustrations of particular cases and examples; and it is hoped that the subject will thus be rendered intelligible and attractive to the learner. This

part of arithmetic is too generally neglected, both in treatises on the subject, and in teaching: and thus one of the principal divisions of mathematical science is converted into a mere practical art; and what is peculiarly fitted to call forth and improve the reasoning powers, is degraded into a dry exercise of memory. It will be found also, that, besides acquiring intelligence and habits of inquiry, so useful throughout life, the pupil who is taught to understand the reasons of the various processes, will find the study much more agreeable than if it were pursued in the usual way, and he will become more expert, and will acquire a much larger amount of real and substantial knowledge, than he otherwise would in the same length of time.

Of the examples and exercises, some are proposed in purely abstract terms, being intended merely to afford practice to the learner in the rules and modes of calculation. In addition to these, there are given, in the parts of the work in which it could conveniently be done, other questions, which will not only afford the pupil farther exercise on the rules that precede them, but will also furnish him with many important facts in geography, astronomy, chronology, and other departments of knowledge. As the information contained in these questions has all been derived from authentic sources, its correctness may be depended on; and it is hoped, that what is thus presented may excite in the young reader a desire to enrich his mind, by the acquisition of other information of a similar nature.

The principal and more important parts of the work have been explained and exemplified more largely than the rest; and in these parts a greater number of exercises have been left unwrought for the improvement of the pupil. Thus, the simple and compound rules, proportion, practice, interest, and exchange, have been treated at considerable length; and much care has been taken to render the operations employed in them as

simple and easy as possible. Specimens of the most common and useful kinds of merchants' accounts, such as invoices, sales, accounts current, &c., according to modern and approved forms, are likewise given, and will be found to form useful exercises for pupils who may be intended for the counting-house.

Some subjects are omitted, which in several works on arithmetic are given at considerable length. Such is barter—an application of the rule of proportion, which is of scarcely any use as a part of mercantile arithmetic. Neither has the method of calculating annuities at simple interest been introduced; as it is unjust in principle, and, in real transactions, all calculations regarding annuities are made according to the principles of compound interest. Many applications of the rules to the circumstances of bodies in motion, and to other subjects in natural philosophy which have been given in works on this science, have also been omitted, as the mere arithmetical pupil cannot be acquainted with their principles. Other subjects, however, have been introduced, which are little known, but which cannot fail to be interesting to the more advanced student. Such are continued fractions, the theory of scales of notation, and the article on mental arithmetic. Great improvements have also been introduced in the extraction of roots.

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TREATISE ON ARITHMETIC

IN

THEORY AND PRACTICE.

INTRODUCTORY EXPLANATIONS.

NOTE.—This chapter of Introductory Explanations is meant rather for the consideration of teachers, and for affording suggestions towards verbal explanations from them to their pupils, and for affording information to advanced learners, than as a passage to be continuously studied by young beginners.

ARITHMETIC * is the science of numbers and of quantities expressed and considered numerically.

It has to do with the question *how many*; and it has besides to do with the question *how much*, when the answer is to be by a number or combination of numbers.

It may now be asked, *What is number, or what are numbers?* To this question there is no short reply possible. Notions of numbers are arrived at in many ways; and, indeed, by every intelligent mind considerable progress is made in their acquisition even in early childhood. By consideration of a few examples, such as will now be briefly sketched out, the learner may acquire clear fundamental notions respecting the nature of numbers, or may get his existing knowledge extended or confirmed.

If we put a pebble into an empty basket, we say there is *one* pebble in it. If we put another in, we say there are now *two* in it, or we say that the *number* of the pebbles in the basket is *two*. If we put again another in, we say there are *three* in it, or we say that the *number* of the pebbles is *three*. If, at the same time, a farmer puts a sheep into a field where there was none before, we say there is *one* sheep in the field. If he puts another in, we say there are *two* in the field; and if he puts again another in, we say the *number* of the sheep in the field is *three*. Now, the group of sheep in the field is very unlike to the group of pebbles in the basket; but, in one respect, the groups can be perceived to be perfectly alike, and that quality or character of likeness is expressed by saying that the

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* The name arithmetic is derived from the Greek word *arithmos*, *number*.

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two groups are alike in number, or are of the same number, or that the *number of the sheep is the same as the number of the pebbles.*

Again, if yet another pebble were put into the basket, and another sheep were put into the field, there would be a new number of pebbles and a new number of sheep, and the two groups so arrived at would be again perceived to be perfectly alike in one character, while quite unlike in every other respect. This character in which sameness can be perceived in the two new groups is still called *number*, and the new number now arrived at, different from the number which was before attained, and was named three, is called the *number four.*

We can after this go on indefinitely, increasing the number of the pebbles, or of the sheep, or of both, by continually adding one more to those already put together; and we can give a new name to each new number arrived at, which is more by one than the previous number.

It is scientifically proper, and it is essential for practical convenience (although sometimes misfitting awkwardly with the arbitrary structure of the English and of many other languages), to call *one* a number, as well as *two*, *three*, *four*, *&c.*; and thus to maintain freedom to say, when only one pebble is in the basket, that *the number of pebbles in it* is one; or freedom to say, that if pebbles are from time to time put in and taken out, so that the number in the basket is to be considered as varying, we may have *the number of the pebbles* in it, for instance, *three*, then *four*, then *two*, and afterwards *one*. This is to the same effect as to speak of a group of things being changed in its number till we have to regard the group as being reduced to only a *single one* of the things which might be brought together into it.

Sometimes also even *zero*, or *nought*, is called a number; but this application of the term "number" can scarcely be supported, unless occasionally for convenience, and with some correction mentally introduced; since 0, nought, or zero, expresses the absence of any one or more of the things contemplated as counted or expressed. As a matter of convenience, however, it is allowable to make such statements as that the numbers of births in a town on five successive days were 3, 1, 2, 0, and 4; and the extension of the signification of the word "number" in this way is scarcely, if at all, liable to introduce any perplexity or any inaccurate modes of thought.

The explanations now given are sufficient to show fully what is the character of groups of objects which can properly be called their number, and to specify the different characters which can properly be called different numbers, and which are distinguishable by different names, as one, two, three, four, five, six, seven, ... eleven, ... seventeen, ... twenty-four, &c.

Besides numbers, properly so called, there are very important numerical expressions, which may be called *fractional numerical expressions*, such as *one half*, *two thirds*, *four and three fifths*, *seven and two tenths and three hundredths*—or, as they may also be written, $\frac{1}{2}$, $\frac{2}{3}$, $4\frac{3}{5}$, 7·23 — to which the designation *number* is frequently applied. The application of the word "number" to such fractional numerical expressions, although sanctioned by very frequent usage,

involves, unfortunately, ambiguity in language, and tends to introduce perplexity in thought. It is indeed essentially inconsistent with the employment of the word "number" in the simple sense which has been already explained, and which is certainly sanctioned by universal usage as the primary or fundamental meaning of the word. Fractional numerical expressions have no direct applicability in counting the number of sheep driven into a field, or of rough lumps of broken stones which might be put into a basket, and which, though different in their sizes, and in their forms, and in their material—some being perhaps of basalt, some of flint, and some of slaty rock—might still be perfectly well used as counters to indicate the number of sheep put into the field. A living sheep can only be counted properly as one: it cannot be regarded as made up of three third parts of a sheep, each alike with the other two. Also we cannot put $4\frac{1}{2}$ rough fragments of stone into a basket; if we put four in, and then, for the half, break another stone into two parts, and put one of those parts into the basket, we shall find that we have really put five pieces of stone into the basket. But otherwise and indirectly fractional numerical expressions may be used for expressing numbers of individual objects by the device of using groups of those objects and fractional parts of a group, as when we speak of $4\frac{1}{2}$ dozens of eggs, or $5\frac{3}{4}$ millions of persons; but in each case here we state directly, not the number of individuals, but the number of groups, and we superadd a fraction of a group; and by custom we often slip into speaking of the number and the fraction jointly as being *a fractional number*. Again, though, as we have seen already, we cannot put four and a half rough fragments of stone into a basket, yet we can perfectly well put four pounds of butter and half a pound of butter, or what for brevity is called four and a half pounds of butter, into a basket. We here meet with a very important distinction in passing from the counting of individual objects, such as fragments of stones, to the numerical statement of quantity of any kind of indefinitely divisible thing, such, for instance, as quantity of butter. We are passing from the one province of arithmetic which has to do with the question *how many*, and entering on the other province which has to do with the question *how much*. It is only in giving numerical expression to the *muchness* or *quantity* of things that fractional numerical expressions unavoidably arise; they have no applicability in expressing directly the *numerousness* or the number of individual objects, though indirectly they can be used for making known the number of individual objects by taking the objects in groups, and then stating, not the number of the objects themselves, but the number of groups and the fractional part of another group, as when we say 5 millions of persons and $\frac{3}{4}$ of another million, or briefly $5\frac{3}{4}$ millions of persons.

Fractional numerical expressions are referred to at this place merely to notice them in connection with the meaning of the word "number," and to make early mention that they are distinct from numbers properly so called, and to state that though they are often spoken of as numbers, yet this arises through imperfections of the language which has been handed down to us from remote times, imperfections which the people at any period living in the world have

not power completely to amend.* The meaning of these remarks about fractions will be more fully appreciated by the learner when he comes to study the explanations of fractions which will follow at various places in this treatise.

It is proper to notice that *things* which can be counted or characterized by number, through the statement of how many there are of them, are not necessarily tangible objects of bodily substance, such, for instance, as pebbles, sheep, houses, &c. The word *thing* may be taken in a very general sense to include any *article, event, extent of time, extent of space*, or other *object of our conception* which can be considered by itself, and also as associated with others like itself. Then a single *thing* is sometimes spoken of as ONE of its kind of things, and sometimes it is called a UNIT. A group or assemblage of any of those single things, as, for instance, a dozen of them or a million of them, is itself also an object of our conception; and may, when we please, be treated as a *single thing*, or as a *unit*, of which we can count two, or three, or five, or any other number. The treating of a number of objects as a "unit," meaning *one group*, and then expressing by a *number* how many there are of those *ones* or single groups, is a process of thought which has very important applications in arithmetic, especially in fractional arithmetic.

Now for giving numerical expression to quantities of things, as, for instance, to quantities of money, of weight, of time, or of length, a particular quantity of the thing to be dealt with is in some way specified, and is called a UNIT of that thing; and to that *unit* some *name*, or *denomination*, is usually given. Thus a pound sterling is a unit of money, an ounce is a unit of weight, an hour is a unit of time, and a foot is a unit of length. The selection of the unit in

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* As one step towards abating the difficulty and perplexity of language here referred to, and also for the sake of brevity, we may, when we find it convenient, call a fractional numerical expression by the shorter name a *fractional numeric*; and we may use the name *numeric* to signify numerical expressions, whether numbers properly so called, such as *one, two, three, four,.... eleven,.... seventeen, &c.*, or fractional numerical expressions, such as $\frac{1}{2}$, $\frac{2}{3}$, $4\frac{3}{5}$, 7·23, &c. This nomenclature will at once supply us with (what is at present an important desideratum) a word free from ambiguity for what is implied by "number" when used in its extended sense, or a single word which will briefly and precisely express what is often designated habitually by the inconvenient name in four words, "*whole number or fraction.*" Further, it would, if it were once sufficiently established in use, allow of our practically keeping the word "number" for its own proper signification, saving us from being obliged to use or interpret it in some cases as restricted to its proper signification, and in other cases as extended so as to include fractional numerical expressions. It is difficult, however, to make any considerable change in language which is already established in customary use, and in the present treatise no complete reform will be attempted. The effort will be to teach almost exclusively the ordinary nomenclature in common use, and which ought to be made known to all learners of arithmetic in present times; but still every effort will be made to use to the best advantage the best parts of the ordinary modes of nomenclature, and, as far as possible, to avoid or to amend the more faulty parts. In this way we must still be contented sometimes to bear with more or less of the old incongruities and ambiguities in verbal statements; and we must be ready to make adjustments or amendments by mental reservations or by temporary explanations. If a pupil were taught arithmetic from a book in which all objectionable or faulty nomenclature was completely obviated, he might sometimes be at a disadvantage in answering in competitive examinations to examiners using ordinary language, defective perhaps in precision or in perspicuity, and might often in ordinary intercourse with other people be insufficiently prepared for ready communication of ideas.

which a quantity of anything is to be expressed is arbitrary; and usually there are several units established in which quantities of a thing may be numerically expressed. Thus time may be expressed in years, in minutes, or in seconds, taken as units, as well as in hours; weight may be expressed in tons, in hundredweights, or in pounds, as units, as well as in ounces; and so on. Sometimes one unit may be much more convenient than another for some particular purpose. Thus it may be more convenient to express a weight of coals as so many tons than as so many ounces; and it may be convenient to express a quantity of tea in pounds as units, and very inconvenient to express it in tons, or as a fraction of a ton, in such a way as will soon be explained.

The unit having been selected, the quantity required to be expressed is specified by stating the number of the selected units which would make up the quantity, if any number of them would exactly do so; or, if not, then by stating the number of them, and the fraction or portion of an additional unit which would make up the quantity. One of the usual modes by which the magnitude of the fraction or additional part of a unit is specified, either quite exactly or with sufficient nearness to the truth, is by conceiving the selected unit to be divided into any convenient number of equal parts, and taking one of these parts as a *subordinate unit* by which to measure the fractional part; the quantity of that fractional part being stated exactly, or approximately, by telling the number of the *subordinate units* that it contains. If, for instance, in order to measure the length of an object, we have selected the inch as the primary unit, and an eighth of an inch as the subordinate unit, and if we find the length of the object to be 7 *inches* and 3 *eighths of an inch*, we have got the length specified numerically through means of a known unit of length, the inch. We may, however, regard the result in several different ways. One way is to adhere to the mode of thought by which we have arrived at it, and so to consider it to be a numerical expression consisting of *two distinct numbers*, 7 of one unit and 3 of a smaller unit of the same kind of thing—namely, length. Another way is to regard it as being 7 *inches, and an eighth of a length of three inches*; so that in this way no mention nor thought of the subordinate unit, before dealt with, is introduced at all. In both cases the fractional part of an inch, required with the 7 inches to make up the whole length, is specified by means of three *components*—namely, the numbers 3 and 8, and the name of the selected unit of length, *the inch*—and so it is denoted as $\frac{3}{8}$ *inch*. The numerical expression $\frac{3}{8}$ in this is called *a fraction*, and is sometimes more particularly described as an *arithmetical fraction*. The whole length then comes to be expressed as 7 inches and $\frac{3}{8}$ inch, or more briefly as $7\frac{3}{8}$ inches. Then as, in speaking of 5 inches, we say the number of inches is 5; or as, in speaking of 10 inches, we say the number of inches is 10; so, in speaking of $7\frac{3}{8}$ inches, we very naturally slip into saying that the *number* of inches is $7\frac{3}{8}$. The occasional extension of application of the word "number" to fractional expressions in this way, while its primary and specially proper signification is also retained, and while statements about numbers are often made which do not hold good for fractional

numerical expressions,* has been noticed already as tending to ambiguity and perplexity. On the other hand, there are important advantages of convenience and of brevity in our agreeing to speak of the *number* of inches in a given length as being $6\frac{5}{8}$, or of the *number* of gallons required to fill a cistern as being $5\frac{4}{10}$, and in speaking often of fractional numerical expressions as being *numbers*. In fact, with the existing language available to us, we cannot avoid often using the word number in this extended way. Sometimes, for distinction, the true or proper numbers, such as 3, 4, 5, 17, 22, &c., are called *exact numbers*, *whole numbers*, or *integers* (the word *integer* being just the Latin word for *whole*), and they may well be called *proper numbers*; while such expressions as $\frac{1}{2}$, $\frac{2}{3}$, $7\frac{4}{5}$, &c., are called *fractions*, or *fractional numbers*, or *fractional numerical expressions*; but for brevity, or through inadvertence, such distinctive names are commonly neglected. Now, there is clearly a departure of our language from adaptation to any precise idea when we talk of a *fractional number of times a complete unit*, or of *a complete unit taken a fractional number of times*; and instead of trying to bring out a clear notion from those words themselves, we are just to bear in mind that they are to be interpreted as meaning either *a fractional part of a complete unit*, as in the case of $\frac{1}{2}$ or $\frac{2}{3}$, or else *a number of the units and a fractional part of another of them*, as in the case of $7\frac{4}{5}$.

Often in practice a quantity of any kind of thing is expressed

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* In support of this assertion two instances may be cited, one exemplifying usage in practical business affairs, and the other exemplifying usage among scientific writers on arithmetic. These two may suffice, but the list might easily be extended indefinitely.

In the public regulations for Post Office Savings Banks, issued by authority of the Postmaster-General, the intimation is made that "*Deposits of one shilling, or of any number of shillings, or of pounds and shillings, may be made by any person at the Post Office Savings Banks,*" subject to some provisions which need not be considered here, being merely for assigning limits to the amounts which will be accepted as deposits from one person. Now, the words here quoted would convey a false statement of what the Post Office authorities really mean to announce, if the word "number" were allowed to mean a fractional numerical expression. The announcement is obviously framed on the presumption that the word "number" in it can only mean *legally* what in the present treatise in the text above is stated as its only *proper* signification—namely, what is commonly designated as "a whole number;" as, for instance, 3, 4, 5,....16,....22, and the like; but not $2\frac{1}{2}$, $3\frac{3}{4}$, 5·17, or the like. If an intending depositor, understanding the word "number" in the extended sense in which it is very often and also quite authoritatively used, would offer a deposit of $3\frac{5}{8}$ pounds, his offer would be refused, as it would amount to 3*l.* 12*s.* 6*d.*, which is not contemplated in the regulations as an amount to be accepted as a deposit.

Again, in the treatise on arithmetic by Professor De Morgan, an author justly recognised as of high authority in arithmetical science, we find the statement (in the chapter on Division, § 99, at page 48 of the 5th edition, 1846) that "*All numbers are measured by* 1; *that is, contain an exact number of units.*" This statement, again, only holds good if the application of the word "number" to fractional numerical expressions is excluded. Throughout his treatise on arithmetic, indeed, he seems generally to have been very careful to avoid applying the name "number" to fractional numerical expressions. These he generally calls fractions, whether they be greater or less than *one*. He usually speaks of "numbers" and of "whole numbers" as synonymous. But still occasionally he seems to slip, as others habitually do, into speaking of a fractional numerical expression under the name "a number," as, for instance, where (at page 82, § 151, of his 5th edition) he speaks of a number of miles as having been measured and found to be 17·846217 miles.

by a number of units of it, each of one magnitude; together with another number of units, each of a smaller magnitude; and perhaps with other numbers of units of still smaller magnitudes added. Thus a quantity of money may be expressed by a certain number of pounds, together with a certain number of shillings, and a certain number of pence, as when we state a quantity of money by the designation £26-15-11; and a period of time may be expressed as 5 hours, 23 minutes, and 18 seconds. Now, the expression £26-15-11 may properly be called a numerical expression for a certain quantity of money; and that numerical expression may properly be said to be made up of three numbers of three distinct units.* Any such expression is called a *compound expression*, and the quantity expressed is often called a *compound quantity*. The quantity so expressed is also spoken of as being *expressed in units of different magnitudes*, or *expressed in units of different denominations*, or, briefly, as being *a quantity in different denominations*; and the expression is said *to comprise different units*. When a quantity is expressed by a number of units of it, each of one magnitude, it is sometimes spoken of as *a simple quantity*, or a *quantity in one denomination*, or a quantity expressed *in units of a single denomination*, or, briefly, *a quantity expressed in one unit*.

A number is said to be *abstractly considered* when it is considered as disassociated from any particular objects or units which it might count or characterize numerically, and as left applicable to things in general. For instance, when we say twice three are six, or four and three are seven, or 5 is contained 3 times in 15, the numbers are all *abstractly considered*. On the other hand, a number is said to be *concretely considered* when it is considered in connexion with some particular objects, or particular units of quantity, which it counts or characterizes numerically. For instance, when we speak of 7 apples, 5 horses, 3 ounces, or 8 yards, the numbers mentioned are all concretely considered. *Numbers abstractly considered* are often called *abstract numbers*, while *numbers concretely considered* are, on the other hand, often called *concrete numbers*. It ought not, however, to be imagined that there are really two different kinds of numbers, abstract and concrete; the number in either case is purely *a number*, and not one kind of number out of more kinds than one. The whole expression *five horses* means a certain group of horses—a group characterized by the number five—and the *five* in that expression is purely a number. Also the whole expression *eight yards* is not a *number*, but is a *quantity of length*, or is *the expression for a quantity of length*. The 8 in that expression is purely a number; and if called a concrete number, the concreteness is merely in our employment of it as connected with the yard unit, so that the two come to denote jointly a quantity of length.

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* The practice, which is rather common, of calling such an expression as £26-15-11 *a number* is objectionable, and can only be supported by some rather strained views of the nature of the expression; as, for instance, by saying that the 15 in it means 15 times 12 of the penny units, and that 26 means 26 times 20 times 12 of the penny units, and that so the whole expression denotes in a complicated way a certain number of the penny units. The simple state of the case is that the expression denotes 26 of the pound units of money, 15 of the shilling units, and 11 of the penny units; and so consists of *three distinct numbers*.

It is important now to observe that fractions in arithmetic may, like numbers properly so called, be considered either abstractly or concretely. If we speak of $\frac{5}{8}$ *of an inch*, or $\frac{5}{8}$ *of a gallon*, or $\frac{5}{8}$ *of a yard*, the numerical expression $\frac{5}{8}$, occurring in each of these three entire expressions, is an arithmetical fraction, applicable alike to the inch unit, the yard unit, or any other unit of any other kind of thing measurable in quantity. We could speak just as well of $\frac{5}{8}$ of a pound of matter, or $\frac{5}{8}$ of a certain unit of power in steam-engines, water-mills, &c., called a horse-power. Thus the fraction $\frac{5}{8}$ may be spoken of and dealt with as disassociated from any particular kind of unit, but left applicable to units of quantity in general; and in this way it is *abstractly considered.* On the other hand, the whole expression *five eighths of an inch* means a quantity of length, and that length may be called a fraction of an inch. That quantity of length may be exhibited to the eye, or submitted to the touch, on an ordinary foot-rule, or by a little piece of wire cut of that length. The piece of wire would show and preserve the length intended, without any reference to the inch unit being required at all; but that same length, when specified by a numerical fraction $\frac{5}{8}$ conjointly with the inch unit, in being named $\frac{5}{8}$ of an inch, comes to be called a fraction of an inch. In this case the arithmetical fraction, being used conjointly with the inch unit to specify a quantity of length, may be said to be *concretely considered*; and corresponding statements might be made in respect to $\frac{5}{8}$ of a gallon, $\frac{5}{8}$ of a pound of matter, or $\frac{5}{8}$ of any unit of any other kind of thing measurable in quantity.

Throughout the foregoing explanations the word *quantity* has been applied in its proper sense, in which it is quite distinct from *number*, and signifies *muchness*, or signifies what is contemplated under the question *how much?* In this proper sense *a great quantity* of anything would mean *much* of that thing, while *a great number* of any things would mean *many* of those things. The word "quantity," however, is often extended in its signification by being taken to serve indiscriminately as a designation both for number and for quantity properly so called. This occasional extension of meaning of the term is subject to the great objection of tending to ambiguity of expression and to confusion of ideas, but still it is so frequent in the customary language of arithmetic, and more especially in that of algebra, that it ought to be known and understood. Indeed, the usage is so completely established in algebra that it can scarcely now be entirely discontinued; but in all cases of its employment care should be taken to have the intended meaning sufficiently shown by the writer and properly understood by the reader. In respect to the correct uses of the words "quantity" and "number," it may be remarked that we cannot properly speak of a quantity of water as equal to a quantity of length or to a quantity of time, but that we can properly speak of equality between the number expressing a quantity of water, and the number expressing a quantity of length, and the number expressing a quantity or period of time.

NUMERATION AND NOTATION.

NUMBERS are usually expressed either by words or by signs or characters. The whole subject of the expressing of numbers is properly called NUMERATION, and the part of it which relates to the expression of numbers by characters is called NOTATION.*

In modern arithmetic all numbers are expressed by means of ten characters, or figures, called the ten digits,† which are shown here following: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. The first of these, called the cipher, or zero, or nought, indicates a place kept void from any other figure, and has important significance through the effects which it is capable of producing on the values or importances of other figures placed in connexion with it, as will soon be explained. The remaining nine denote respectively the numbers one, two, three, four, five, six, seven, eight, nine. These nine are commonly called the *significant figures*; but this designation ought to be abandoned as conveying essentially an erroneous idea, since the figure 0, in expressing precisely the absence of the thing or things (or of the number applicable to things in general) that would be expressed by any other figure put instead of it, has quite as definite a signification as any of the other nine. When a distinctive appellation is wanted, the figures 1, 2, 3, 4, 5, 6, 7, 8, 9, may be called the *value figures*, and nought, 0, the *void figure*.

When any of the nine "significant" or value figures stands by itself, or when it is followed by no other figure, it expresses merely its number of *ones*, or of single things counted; but when it is followed by one figure it expresses its number of *tens* of the things counted; when by *two*, it expresses its number of *hundreds* of them; when by three, it expresses its number of *thousands* of them; and so on. Thus in 3333 each 3 stands for *three* of something; the first 3 at the right means three of the things counted; the second signifies 3 groups of ten each; the third, 3 groups of a hundred each; and the fourth 3 groups of a thousand each.

Thus in the expressions 5, 25, and 365 the figure 5 denotes simply five of the things counted; while in 52 and 256 it means five tens of the things counted, or fifty of them; and in 524 it denotes five hundreds of them. Thus the expression 576 means five hundreds, seven tens, and six units; or, as it is briefly read, five hundred and seventy-six.

When a number expressed by several figures is to be read off in words, the three next the right hand, taken together, are read as

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* The word *numeration* is frequently used, in a restricted sense, as meaning merely the method of expressing in words the values of numbers already expressed by characters; while, in conjunction with that restricted signification, the word *notation* is applied to the reverse process of expressing by characters numbers already given in words, or else to the process of expressing them by characters when they are given in any way; as, for instance, when a group of objects is given to be counted, and the number is required to be *noted* in characters. It is better to apply the name numeration to the whole subject than to restrict it to one part.

† The name digit is derived from the Latin word *digitus*, finger. The ten figures used in arithmetic are by some called the zero and nine digits, and by others they are called in preference the ten digits.

being a number of units; the three next them, taken together, are read as being a number of thousands; the next three are read as being a number of millions; and so on, according to the subjoined table. When a line of figures is thus divided, the three next the right hand are called the *first period*, the next three the *second period*; and so on, as in the table.

NUMERATION TABLE.

&c.	&c.	
VI. Quadrillions ...	Hundreds of quadrillions Tens of quadrillions Quadrillions	18, 17, 16
V. Trillions	Hundreds of trillions Tens of trillions Trillions	15, 14, 13
IV. Billions	Hundreds of billions Tens of billions Billions	12, 11, 10
III. Millions	Hundreds of millions Tens of millions Millions	9, 8, 7
II. Thousands ...	Hundreds of thousands Tens of thousands Thousands	6, 5, 4
I. Units	Hundreds Tens Units	3, 2, 1

The periods after quadrillions may be called quintillions, sextillions, septillions, octillions, and nonillions; and analogical names might be formed for the still higher periods. In actual practice, however, it is seldom necessary to *name* numbers exceeding millions.

The *local value* of any figure used in expressing a number, is at once discovered from this table. Thus, 6 in the eighth place from the right hand, expresses six tens of millions, or sixty millions; and, conversely, sixty millions will be expressed by the figure 6 in the eighth place.*

* The method given above of dividing lines of figures into periods, and of naming those periods, is that which is employed by the French and Italians. It is strongly recommended by its simplicity and elegance; and it has been adopted in some works in this country. In most English works, however, the periods have been made to consist of *six figures* each; and they have been designated by the same names as those in the table given above, except thousands, for which there is not a distinct period. The two methods agree as far as hundreds of millions, and it is rarely necessary to name larger numbers. As the old method is still taught in many English books, and in order to warn the pupil that he must be careful to understand the words used for the expression of large numbers in the sense intended by the speaker or writer, the old method is shown in the subjoined table; and the answers of the exercises are given according to both methods at the end of the book. In France a milliard is the same as what is called a billion in the foregoing table; it is a thousand millions. It is scarcely necessary to say, that the rules and directions given in the text will be applicable in this method, if the periods be made to consist of six figures each, instead of

absence of value, is used in com-
where no value is to be expressed,
res occupy those places in which
lues.

this manner, denote three hundred and
, which contains the same value figures,
ns of thousands, six thousand, no hundreds,
ee hundred and six thousand, and fifty.

e explained, we have the following
ns in numeration which are generally

n words the numbers denoted by Commencing at the right-hand
gures into periods of three figures
than three remain. (2.) Then,
t side, annex to the value expressed
h period, except that of the units,
riod, according to the numeration

, 37053907, becomes by division into periods
d *thirty-seven millions, fifty-three thousand,*
n, the term *units*, or *ones*, at the last being

period be called millions, the third billions, &c., as in

OLD NUMERATION TABLE.

III. Billions	7	Thousands of billions
	16	Hundreds of billions
	15	Tens of billions
	14	Billions
	13	
II. Millions	12	Hundreds of thousands of millions
	11	Tens of thousands of millions
	10	Thousands of millions
	9	Hundreds of millions
	8	Tens of millions
	7	Millions
nits	6	Hundreds of thousands
	5	Tens of thousands
	4	Thousands
	3	Hundreds
	2	Tens
	1	Units

By practice the pupil will soon find it unnecessary to divide into periods any lines of figures, except those of considerable magnitude.

Exercises in converting the Expression of Numbers from Figures to Words.

Write down in words, or name, the numbers signified by the following expressions: —

Exer. 1. 24	Ex. 4. 1000	Ex. 7. 9790	Ex. 10. 4055070
2. 144	5. 1728	8. 37048	11. 300405
3. 365	6. 2240	9. 30009	12. 79503040

Ex. 13. 800560080	Ex. 22. 100000001000
14. 57290000	23. 60660607007
15. 680000042	24. 1020304050607
16. 93090093	25. 910110120301
17. 113355	26. 200030040538
18. 785398	27. 820760005192645
19. 7030462	28. 10010001000100
20. 24902490.	29. 40506070089000
21. 9003008005	30. 794628900640030004

RULE II. *To express numbers by figures*: (1.) Make a sufficient number of ciphers or dots, and divide them into periods of three each. (2.) Then, commencing at the left, place, in their proper positions, beneath the dots or ciphers, the value figures necessary for expressing the proposed number. (3.) If any places remain unoccupied, let them be filled with ciphers.

Thus, the method of expressing the number, two hundred and five millions, twenty thousand, seven hundred, and nine, will be found in the following manner :—

000,000,000, or . . ., . . ., . . .,
2 5 2 7 9, 2 5 2 7 9;

and thence, by filling the unoccupied places, 205,020,709.

By practice the learner will soon be able to dispense with the use of the dots or ciphers.

Exercises in converting the Expression of Numbers from Words to Figures.

Express the following numbers in figures :—

Exer. 1. Fifty-two.
2. Three hundred.
3. Five hundred and four.
4. One thousand and twenty-four.
5. Two thousand.
6. One thousand, eight hundred, and fifteen.

Exer. 7. Seven thousand, eight hundred, and fifty-four.
8. Three thousand and eight.
9. Five thousand and seventy.
10. Four thousand, five hundred, and four.
11. Twenty thousand, and eighty-four.
12. Six hundred and fifty thousand, and ninety.
13. Seven millions, seven thousand, and ten.
14. Sixty-four millions, three hundred.
15. Eleven millions, two thousand.
16. One hundred and ten millions, and twenty thousand.
17. One million, and fifty thousand.
18. One billion, two hundred thousand.
19. Seventy billions, ten thousand, and eighty-eight.
20. Nine hundred billions, sixty-eight millions, and twenty.
21. The following numbers express the distances of the principal primary planets from the sun, in miles;—express them in figures:—Mercury, thirty-five millions, five hundred thousand; Venus, sixty-six millions, three hundred thousand; the Earth, ninety-one millions, six hundred thousand; Mars, one hundred and forty millions; Jupiter, four hundred and seventy-six millions; Saturn, eight hundred and seventy-four millions; Uranus, one thousand, seven hundred and sixty millions; Neptune, two thousand, seven hundred and fifty millions.*

Such is the facility with which large numbers are expressed, both by figures and in language, that we have generally a very imperfect conception of their real magnitudes. The following considerations will assist in enlarging the ideas of the pupil on this subject:—To count a million, at the rate of one in each second of time, would require between twenty-three and twenty-four days of twelve hours each. The seconds in six thousand years are less than one fifth of a trillion.†

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* The distance of Uranus might be stated as one billion, seven hundred and sixty millions; and that of Neptune as two billions, seven hundred and fifty millions of miles: but the mode of naming these distances used above is well suited for conveying the information in this case, because here the *million of miles* may conveniently be considered as a unit of distance, and the distances of these planets may be thought of as so many *millions of miles*, instead of being thought of as so many *miles*. In like manner large sums of money are often spoken of in millions of pounds, instead of in pounds. Thus people would often prefer to say two and a half millions of pounds, rather than two millions, five hundred thousand pounds.

† This one-fifth of a trillion of the new numeration would be designated as one-fifth of a billion, in the common or old numeration.

A quadrillion* of leaves of paper, each the two-hundredth part of an inch in thickness, would form a pile, the height of which would be three hundred and thirty times the moon's distance from the earth. Let it be remembered, also, that a million is equal to a thousand repeated a thousand times; and a billion † is equal to a million repeated a thousand times.

In the ancient Roman notation, I. signified one, V. five, X. ten, L. fifty, and C. one hundred. To these characters were added, at a later period, D., signifying five hundred, and M., one thousand. When any character was followed by another of equal, or of less value, the compound value was equal to the simple values of *both taken together*; but when a character preceded one of greater value, both together expressed a value equal to the *difference* of their simple values. Thus, II. expressed two; XI., eleven, and IX., nine; CX., one hundred and ten, and XC., ninety. We find also IƆ. put for 500; and by every such Ɔ annexed, the value is made ten times as great. Thus, IƆƆ. signifies 5000; IƆƆƆ., 50,000, &c. CIƆ. was also used to express 1000, and the prefixing of C and the annexing of Ɔ, increased the value ten times. Thus, CCIƆƆ. signified 10,000; CCCIƆƆƆ., 100,000, &c. A line drawn over a letter, made it signify as many thousands as the letter itself expressed units. Thus, $\overline{\text{V}}$. expressed 5000; $\overline{\text{C}}$., 100,000, &c.

The following table, together with the preceding observations, will give an adequate idea of the Roman notation:—

I.	1	IX.	9	LXXX.	80
II.	2	X.	10	XC.	90
III.	3	XX.	20	C.	100
IV., or IIII.	4	XXX.	30	D., or IƆ.	500
V.	5	XL.	40	M., or CIƆ.	1000
VI.	6	L.	50	MM., or $\overline{\text{II}}$.	2000
VII.	7	LX.	60	IƆƆ., or $\overline{\text{V}}$.	5000
VIII.	8	LXX.	70		

M.DCCC.XLII., or CIƆ.IƆCCC.XLII., 1842.

SIMPLE ADDITION.

THE object of ADDITION, considered generally, is to find the number or quantity which is equivalent to two or more given numbers or quantities taken together.

The number or quantity which is equivalent to two or more given numbers or quantities taken together is called their SUM.

When simply numbers in the proper sense of the term—that is to say, numbers without fractions—are to be added, the operation is called SIMPLE ADDITION.

When *quantities* expressed simply by numbers of units, the units being all of one magnitude, or, as it may be said, all of one denomination and all whole units, as

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* This quadrillion of the new numeration is the same as a thousand billions in the old numeration.

† But a billion of the old numeration is equal to a million repeated a million of times.

all yards, or all gallons, are to be added, the arithmetical operation consists simply in the adding of the mere *numbers* which express the *quantities*; and is, in like manner, called SIMPLE ADDITION.

When quantities expressed numerically by units of one denomination together with fractional parts of the same units, or else by various fractional parts of the same units without whole units, are to be added (for instance, if a length of 3 inches and $\frac{5}{8}$ of an inch, and a length of 1 inch and $\frac{6}{10}$ of an inch are to be added together; or else if $\frac{3}{4}$ of an inch and $\frac{2}{5}$ of an inch, and $\frac{1}{3}$ of an inch are to be added together), the operation is not simple, but is somewhat complicated, and is called FRACTIONAL ADDITION, or ADDITION OF FRACTIONS.

When *quantities* numerically expressed in different units, or, what means the same, when *quantities* in the expression of which units of different denominations occur, are to be added, the operation is called COMPOUND ADDITION. The quantities to be added must be all of the same kind, though the magnitudes and denominations of the units in which they are expressed may be different. Thus the quantities to be added may be all money, and so all of the same kind, while they may be expressed by numbers of pounds, numbers of shillings, and numbers of pence; or they may be all time, and so all of one kind, while expressed in numbers of hours, numbers of minutes, and numbers of seconds.

Quantities of different kinds do not admit of being added together in arithmetic. Thus we cannot add together five inches, four gallons, and three minutes, to find either how many, or how much, of any kind of thing there would be.

The sign +, called *plus*, is employed in arithmetic and other parts of mathematics, to signify that the numbers or quantities between which it is placed, are to be added together; and the sign =, called *the sign of equality*, is used to denote, that the numbers or quantities between which it stands, are equal to one another. Thus, the expression, $12 + 9 = 21$, which is read, 12 *plus* 9 (that is, 12 *more by* 9) equal to 21, means that 12 and 9, added together, amount to 21; or that the sum of 12 and 9 is 21.

RULE FOR SIMPLE ADDITION. *To add numbers or to add quantities expressed by numbers of units all of one magnitude or denomination*: (1.) Place the numbers so that

units may stand under units, tens under tens, &c. (2.) Find the sum of the column of units, set down the last figure of it below that column, and *carry* to the next the number expressed by the remaining figure, or figures, if there be any. (3.) Proceed as before with the remaining columns, and at the last column set down its entire amount.

Thus, to add together 9468, 2956, and 79, let them be set as in the margin, and proceed thus:—9 and 6 are 15, and 8 are 23; set down 3, and carry 2 to the column of tens. Then 2 and 7 are 9, and 5 are 14, and 6 are 20; set down 0, and carry 2: 2 and 9 are 11, and 4 are 15; set down 5, and carry 1: 1 and 2 are 3, and 9 are 12; set down 12: and the sum, or answer required, is twelve thousand, five hundred, and three.

Exam. 1.

$$\begin{array}{r} 9468 \\ 2956 \\ 79 \\ \hline 12503 \end{array}$$ *sum.*

It is to be particularly observed, however, that after the acquirement of a little experience in arithmetic, a pupil, or a person doing addition in actual business, ought to use far fewer words, aloud or mentally, than those which, for the purpose of explanation, are stated in this example. For instance, in adding the second column, counted from right towards left, he ought not to say, "Two and seven are nine, and five are fourteen, and six are twenty," but ought only to say quickly, aloud or mentally, the following words, or even fewer still:—"*Carry two, nine, fourteen, twenty.*" He should then set down 0 and go on with carrying two to the next column. Advice of the same kind might be given in reference to many other examples for explaining arithmetical processes in what follows throughout this book; but the present explanation may suffice for guarding the pupil against a practice which sometimes interferes with attainment of quickness and ease in arithmetical work.

Reason of the Rule.

The rule for performing addition depends on the nature of notation, and on the obvious principle, that *the whole is equal to the sum of all its parts.* By placing units under units, tens under tens, &c., we are enabled the more easily to add together the figures of the corresponding local values; and *one* is carried for every *ten*, because, by the nature of notation, *ten* in any column is equivalent only to *one* in the column immediately to the left of it. We commence with the units merely for the convenience of carrying to the next columns.

Thus, in the preceding example, the sum of the column of units is 23; and therefore, after setting down 3, we have 20 remaining. But, by the nature of notation, 2 in the next column is equal to 20 in this; and therefore we carry only 2.

Some teachers may, perhaps, consider it proper to make pupils commit the following table to memory:—

ADDITION TABLE.

2 and	3 and	4 and	5 and	6 and	7 and	8 and	9 and
2 are 4	2 are 5	2 are 6	2 are 7	2 are 8	2 are 9	2 are 10	2 are 11
3 .. 5	3 .. 6	3 .. 7	3 .. 8	3 .. 9	3 .. 10	3 .. 11	3 .. 12
4 .. 6	4 .. 7	4 .. 8	4 .. 9	4 .. 10	4 .. 11	4 .. 12	4 .. 13
5 .. 7	5 .. 8	5 .. 9	5 .. 10	5 .. 11	5 .. 12	5 .. 13	5 .. 14
6 .. 8	6 .. 9	6 .. 10	6 .. 11	6 .. 12	6 .. 13	6 .. 14	6 .. 15
7 .. 9	7 .. 10	7 .. 11	7 .. 12	7 .. 13	7 .. 14	7 .. 15	7 .. 16
8 .. 10	8 .. 11	8 .. 12	8 .. 13	8 .. 14	8 .. 15	8 .. 16	8 .. 17
9 .. 11	9 .. 12	9 .. 13	9 .. 14	9 .. 15	9 .. 16	9 .. 17	9 .. 18

To enable the learner to acquire accuracy and despatch in addition, he ought to accustom himself to add in the following manner, till he can do it with facility :—Since 6 and 6 are 12, 26 and 6 are 32 : (here he should observe that 12 and 32 end in the same figure :) since 9 and 7 are 16, 39 and 7 are 46 : since 8 and 6 are 14, 88 and 6 are 94 : since 6 and 9 are 15, 36 and 9 are 45, &c. He ought, however, soon to become so expert as scarcely ever to require the aid of such modes of thinking; as he ought to perceive the results momentarily without waiting to think.

Methods of Proof.

1. If the addition has been performed in the order shown in the foregoing example—that is to say, by commencing at the bottom of each column, and proceeding upward—perform the addition anew, but in the reverse order, commencing now at the top of each column, and proceeding downward; and if the result is the same in both cases, it may be presumed to be correct.

2. Separate the given numbers into two or more portions. Find the sums of these portions, severally, and add these partial sums together. If the last result be equal to that found by the common method, the work is right.

This will appear obvious from the following example : —

Exam. 2.	37928
	93640
	23574
	75849
Entire sum . .	230991
Sum of the 1st portion	131568
„ 2nd „	99423
Entire sum . .	230991, *proof.*

This method may also be employed with advantage in finding the sums of large columns, instead of adding the whole at a single operation. The first method of proof answers well, when the number of lines to be added is not great.

3. Commencing at the left hand, add the several columns* *without carrying*, and set down the full sum of each column with the units in their proper place, and the tens below the figure immediately

* The words *add the several columns* are used as an abbreviation signifying *add together the figures in the columns severally.*

to the left. Add together the lines thus resulting, and if the last result agree with that obtained by the common method, it may be concluded that both are right.

Thus, in the annexed example, the sum of the left-hand column is 25, which is set down in full: the sum in the next column is 30; the cipher is set in its proper place, and 3 under the 5; and so with the rest. The sum of the two lines thus obtained, is equal to the sum found by the ordinary method.

This method of addition might be used instead of the common method; and as it requires nothing to be carried, it may be employed with advantage when the calculator is liable to interruptions.

Exam. 3.

```
 5946
 9738
 2697
 9868
─────
28249, sum.
─────
25029
 322
─────
28249, proof.
```

Exercises.

1.	2.	3.	4.	5.
3789	92864	486759	17896	258111
4236	79784	537192	570937	4174456
7483	4759	468013	784947	6880921
6047	28936	16975	9678	9911604

6. 94753 + 2847 + 793688 + 9386 + 258 + 3456.

7. 8289364 + 275224 + 6875144 + 12897 + 7650368 + 94986347 + 42682 + 3749286 + 7676.

8. 294796 + 489276 + 16759284 + 4938 + 5713245 + 3348675 + 798426 + 9482 + 39867.

9. 27515436 + 8937549 + 37246375 + 48795 + 378 + 2863487 + 864937 + 3894 + 7863927 + 826957.

10. 986759 + 4976346 + 29483 + 898647 + 3984753 + 6489778 + 57893 + 2468144 + 576989 + 498653.

11. 4683795 + 248675931 + 94986473 + 2849758 + 53388336 + 7788995 + 2137485 + 6758927 + 4926431 + 27729512 + 7842634 + 949867 + 343216 + 78934.

12. Add together 9466495, 375573735, 754547, 3789284, 29886799, 992984, 293675, 2684487, 3592873, 8847599, 738873, 7849376.

13. Required the sum of 978, 749, 4764, 8967, 79889294, 7759286, 939723, 864937, 99375847, 294886, 94623, 924086, 794867, 935279423, 9738413208, and 2468975.

14. 28674 + 39257 + 3834 + 92751 + 92503 + 86759 + 394875 + 34938 + 375396 + 759394 + 267934 + 6846 + 94657835.

15. Add together seven thousand and ninety-four; two thousand, one hundred, and nine; eight thousand, nine hundred, and sixty; eighty-seven thousand and sixty-two; three hundred and seventy-five; nine thousand and thirty; thirty thousand and forty-six; fifty-four thousand, seven hundred, and seventy-five; seven thousand, eight hundred, and fifty-four.

16. James Watt was born in the year 1736, and died in his

eighty-fourth year. Find two years in one of which, according to this information, he must have died.*

17. William the Conqueror began his reign in England in the year 1066, and reigned 21 years; William II. reigned 13 years; Henry I., 15 years; Stephen, 39 years; Henry II., 35 years; Richard I., 10 years; John, 17 years; Henry III., 56 years; Edward I., 35 years; Edward II., 20 years; Edward III., 50 years; and Richard II., 22 years: in what year did this last prince cease to reign?

18. Captain Cook, in his first voyage round the world, sailed from Portsmouth to the Madeiras, a distance of 1451 British miles; thence to the Canaries, 339 miles; from these to the Cape Verd Islands, 985 miles; and thence to Rio Janeiro, 3058 miles; from that town to Cape Horn, 2659 miles, and thence to Otaheite, 4919 miles; from Otaheite to the most southern point of the voyage, 1619 miles; and thence to Cook's Strait in New Zealand, 1988 miles; from Cook's Strait to Green Cape in New Holland, 1368 miles; and thence along the eastern coast of New Holland to the most northern point of that island, 2176 miles; thence to the straits of Sunda, 2487 miles; and thence to the Cape of Good Hope, 5818 miles; from that cape to St. Helena, 1884 miles; and thence to Ascension Island, 822 miles; from Ascension to Corvo in the Azores, 3462 miles; and thence to Portsmouth, 1598 miles. What was the length of the whole voyage, exclusive of numerous deviations from these courses?

SIMPLE SUBTRACTION.

THE object of SUBTRACTION, considered generally, is to find the number which remains when one number is taken from another; or to find the quantity which remains when one quantity numerically expressed is taken from another.

The number or quantity found as the result in subtraction is called the REMAINDER, the DIFFERENCE, or the EXCESS.†

SUBTRACTION is distinguished into three various cases, SIMPLE SUBTRACTION, FRACTIONAL SUBTRACTION, and COM-

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* You should observe that if you are informed that a person died in his first year, meaning the first year of his life, he may have died in the year of his birth or in the following year, these years being taken as commencing each with a New Year's Day, a 1st of January, while the years of a person's life are taken as commencing each with a birthday. Then, if he died in his second year, he must have died either in the year after that of his birth, or in the next following year, and so on

† The number to be subtracted is sometimes called the *subtrahend*; and that from which it is to be taken, the *minuend*. These terms, however, are properly falling into disuse.

POUND SUBTRACTION; and the distinctions between these may be easily understood by referring to the distinctions already explained (see pp. 14 and 15) between SIMPLE, FRACTIONAL, and COMPOUND ADDITION, with which they are in exact correspondence.

The two given quantities, of which one is to be subtracted from the other, must be of the same kind. Thus, while it is possible to subtract 3 shillings and 2 pence from £4—that is, money from money; or to subtract 7 feet from 4 yards—that is, length from length; it is not possible to subtract 3 shillings and 2 pence from 4 hours.

The sign $-$, called *minus*, when set between two numbers, denotes that the one which follows it, is to be taken from the one which precedes it. Thus, $16-9=7$, which is read 16 *minus* 9 (that is, 16 *less by* 9) *equal to* 7, denotes that if 9 be taken from 16, the remainder is 7.

RULE FOR SIMPLE SUBTRACTION. (1.) Place the less number below the greater,* with units under units, tens under tens, &c., as in addition. (2.) Beginning with the units, take, if possible, each figure in the lower line, from the figure above it, and set down the remainder. (3.) But if any figure in the lower line be greater than the figure above it, add *ten* to the upper; then subtract as before, and carry *one* to the next figure in the lower line.

Methods of Proof.

1. Add the remainder and the less of the given numbers together: if the sum be equal to the greater, the work is correct.

2. Subtract the number found, from the greater of the given numbers: if the remainder be equal to the less, the work is correct.

Exam. 1. From 7854 take 4513.

Set the numbers as in the margin, and proceed thus:—3 from 4, and 1 remains; 1 from 5, and 4 remain; 5 from 8, and 3 remain; 4 from 7, and 3 remain: the remainder, therefore, is 3341.

$$\begin{array}{l} 7854 \\ 4513 \\ \hline 3341, \textit{ remainder.} \\ \hline 7854, \textit{ proof.} \end{array}$$

To prove the work, to the less of the given numbers add the remainder, and the sum is 7854, the greater; or, as in the second method, subtract the remainder from the greater number, and the result is 4513, the less.

$$\begin{array}{l} 7854 \\ 4513 \\ \hline 3341, \textit{ remainder.} \\ \hline 4513, \textit{ proof.} \end{array}$$

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* Though this is the usual method of placing the numbers, the greater is sometimes placed with advantage below the less. In this case, the words *upper* and *lower* must be interchanged throughout the rule. So likewise must *above* and *below*.

Exam. 2. Required the difference of 3712 and 1831.

In this example proceed thus:—1 from 2, and 1 remains: 3 from 11, and 8 remain: carry 1 to 8; then 9 from 17, and 8 remain: carry 1; and then 2 from 3, and 1 remains. The difference, therefore, is 1881; and the operation would be proved in the same manner as before. When we thus add 10, it is commonly said that we *borrow* 10.

$$\begin{array}{r} 3712 \\ 1831 \\ \hline 1881 \end{array}, \textit{ remainder.}$$

Reason of the Rule.

The rule for subtraction depends on the principle, that the differences of the several parts of two numbers are, when taken together, equal to the difference of the numbers themselves. The reason of placing units under units, tens under tens, &c., is, that figures may be subtracted from others of the corresponding local value with more facility. By carrying one to the lower figure, we increase the lower line as much as we increased the upper; and thus the difference is the same as if neither had been increased.

Thus, in the second of the preceding examples, when in the tens' place we subtract 3 from 11, we add 10 to the 1 in the upper line; then the lower line is increased by the same value, by adding 1 to the 8; because, by the nature of notation, 1 in the third column is equivalent to 10 in the second. Thus, therefore, both the given numbers are equally increased, and consequently the difference must be the same as if they had received no increase.

As a further illustration of subtraction, let it be required to find the difference of 83 and 57. Here, as 7 cannot be taken from 3, we may consider 83 as equal to 70 and 13; and subtracting 7 from 13, and 5 from 7, we find the difference to be 26. In this simple and natural method, the values of the given numbers undergo no change; and, with only one exception, it might be employed with as much facility as the common method, the next figure in the upper line being always diminished by a unit, when one would be carried to the figure below it in the common method. The exception is the case in which the next figure in the upper line is 0. In this case the common method is considerably preferable; and, as in practice, that method is in no case inferior, it is universally preferred.

$$\begin{array}{r} 83 \\ 57 \\ \hline 26 \end{array}$$

Exercises.

1. 45079 − 32048	8. 915161718 − 151617189
2. 33456 − 17748	9. 202122223 − 192021222
3. 65934 − 48566	10. 357912468 − 24680135
4. 90401 − 58270	11. 7503046571 − 34992884
5. 623417 − 32686	12. 376995145 − 49490718
6. 8463192 − 177825	13. 153425178 − 53845248
7. 4444444 − 1234567	14. 100000000 − 10001001

15. From a piece of linen 85 yards long, a piece 57 yards long is cut off: find how much remains. Observe: this is a case of sub-

tracting a quantity from a quantity, and the result is to be a quantity, not merely a number.

16. Subtract three thousand from three millions.

17 From a cistern which contained 251 gallons of water, 87 gallons have been drawn off: how much remains in it if none has gone away except those 87 gallons?

18. Required the difference between three, and three hundred thousand.

19. La Place, the celebrated French mathematician and philosopher, was born in 1749, and died in 1827: how long did he live?

20. Mont Blanc, the highest mountain in Europe, is 15,680 feet high; and the height of Chimborazo, the highest in America, is 21,400 feet: how much is the latter higher than the former?

21. The following are the years of the Christian era in which the undermentioned events happened: required the number of years from each till the year 1875. Commencement of the Hegira, or era of the flight of Mohammed, 622; The Arabian or modern notation in arithmetic, introduced from Arabia into Europe by the Saracens, 991; First Crusade, 1096; Magna Charta signed by King John, 1215; Linen first made in England, 1253; Termination of the Crusades, 1291; Gunpowder first used in Europe, 1330; Algebra introduced into Europe from Arabia, 1412; Printing invented, 1440; University of Glasgow founded, 1450; Constantinople taken by the Turks, 1453; America discovered by Columbus, 1492; Copernicus died, 1543; Spanish Armada destroyed, 1588; Telescopes invented, 1590; University of Dublin founded, 1591; Decimal fractions invented, 1602; Logarithms published by Napier, 1614; Barometer invented, 1643; Air Pump invented, 1654; Newtonian Philosophy published, 1686; Union of Great Britain and Ireland, 1801; Battle of Trafalgar, 1805; Battle of Waterloo, 1815; Catholic Emancipation Bill passed by the British Parliament, 1829; Negro Emancipation in the British Colonies, 1838; Corn Laws repealed, 1846; Close of the Crimean War, 1855; Battle of Solferino, 1859; Close of the American War, 1865; Atlantic Cable successfully laid, 1866.

SIMPLE MULTIPLICATION.

THE object of MULTIPLICATION, in the cases to which the term most obviously and properly applies, is to find the sum of a whole number or a fractional numerical expression or a quantity repeated a whole number of times: but the name *multiplication* is extended further to include also cases in which a repetition of the thing is made a whole number of times and a fraction of it is superadded; and also to include cases in which the thing is not repeated at all, nor taken even once, but only a fraction of it is taken.

In SIMPLE MULTIPLICATION, integers, or whole numbers, called also proper numbers, alone are dealt with; the treatment of fractional numerical expressions and of compound numerical expressions not falling within the scope of simple multiplication. We may thus notice that when *quantities* are to be multiplied any number of times, they must be expressed simply in one denomination, and must be without fractions, if the process is to be one in simple multiplication.

The process is called COMPOUND MULTIPLICATION when any quantity expressed in more denominations than one—as, for instance, a quantity of money expressed in pounds, shillings, and pence—is to be conceived as repeated or multiplied a certain number of times, and the amount due to the repetition is to be found.

The number or quantity to be repeated is called the MULTIPLICAND; the number which shows how often the multiplicand is to be repeated is termed the MULTIPLIER; and the number or quantity found is called the PRODUCT. Both the multiplicand and multiplier are sometimes called FACTORS, from their *making* or producing the product.* It will readily be observed that when the multiplier is a whole number, the process of multiplication comes to be merely an abridged method of performing addition in the case in which the numbers or quantities to be added together are equal to one another.

An important principle which holds good for numbers in general, whether whole or fractional, may now be established for whole numbers. It is that the product of two numbers is the same whichever of them is taken as multiplier: that, for instance, 3 times 5 are the same as 5 times 3; that is, that 3 times 5 objects of any kind are the same in number as 5 times 3 objects of any kind. To illustrate this let the objects be represented by dots, and let three rows each containing five dots be placed as in the margin. We have thus 15 objects represented, which may be regarded either as 3 times 5 objects, when we take the three horizontal rows; or as 5 times 3 objects, when we take the five vertical

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. . . . .
. . . . .

* It is plain, from the definition of multiplication, that at least one of the two factors—that is to say, that the multiplier, at least—must be a *mere number, whole or fractional*, not a quantity of any kind of thing; not an expression for money, weight, measure, or any other object. Thus, while we may multiply 5 shillings by 6—that is, repeat 5 shillings 6 times—it would evidently be absurd to speak of multiplying 5 shillings by 6 shillings—that is, of repeating 5 shillings 6 shillings times. Further remarks on this subject will be found in the article on Compound Multiplication.

rows, each of which contains three dots; and a like illustration may be given in every case when whole numbers are dealt with. Fractional numbers and their multiplication will be treated of farther on in this book, but not at this early stage in what is called *simple multiplication.*

The sign ×, called *the sign of multiplication*, placed between two numbers, denotes the multiplication of those numbers together: that is to say, the multiplication of either of them by the other. Thus, 20 × 12 = 240, which, for brevity, may be read, 20 *into* 12 *equal to* 240, denotes that the product of 20 and 12 is 240. When the sign of multiplication × is placed between a number and an expression for a quantity, it denotes the multiplication of the quantity by the number. Thus, 2 × £2 - 3 - 4, or £2 - 3 - 4 × 2, would denote £4 - 6 - 8.

MULTIPLICATION TABLE.*

Twice	3 times	4 times	5 times	6 times	7 times
1 are 2	1 are 3	1 are 4	1 are 5	1 are 6	1 are 7
2 — 4	2 — 6	2 — 8	2 — 10	2 — 12	2 — 14
3 — 6	3 — 9	3 — 12	3 — 15	3 — 18	3 — 21
4 — 8	4 — 12	4 — 16	4 — 20	4 — 24	4 — 28
5 — 10	5 — 15	5 — 20	5 — 25	5 — 30	5 — 35
6 — 12	6 — 18	6 — 24	6 — 30	6 — 36	6 — 42
7 — 14	7 — 21	7 — 28	7 — 35	7 — 42	7 — 49
8 — 16	8 — 24	8 — 32	8 — 40	8 — 48	8 — 56
9 — 18	9 — 27	9 — 36	9 — 45	9 — 54	9 — 63
10 — 20	10 — 30	10 — 40	10 — 50	10 — 60	10 — 70
11 — 22	11 — 33	11 — 44	11 — 55	11 — 66	11 — 77
12 — 24	12 — 36	12 — 48	12 — 60	12 — 72	12 — 84

8 times	9 times	10 times	11 times	12 times
1 are 8	1 are 9	1 are 10	1 are 11	1 are 12
2 — 16	2 — 18	2 — 20	2 — 22	2 — 24
3 — 24	3 — 27	3 — 30	3 — 33	3 — 36
4 — 32	4 — 36	4 — 40	4 — 44	4 — 48
5 — 40	5 — 45	5 — 50	5 — 55	5 — 60
6 — 48	6 — 54	6 — 60	6 — 66	6 — 72
7 — 56	7 — 63	7 — 70	7 — 77	7 — 84
8 — 64	8 — 72	8 — 80	8 — 88	8 — 96
9 — 72	9 — 81	9 — 90	9 — 99	9 — 108
10 — 80	10 — 90	10 — 100	10 — 110	10 — 120
11 — 88	11 — 99	11 — 110	11 — 121	11 — 132
12 — 96	12 — 108	12 — 120	12 — 132	12 — 144

* Though the part of the multiplication table given in the text is enough for the pupil to commit to memory at first; yet, after he has made some proficiency in arithmetic, he may find it advantageous to commit what follows, as it will enable him, in many cases, to shorten his work in a considerable degree. The

It will perhaps enable the pupil to commit the preceding table to memory with more ease if he construct it by addition. Thus, to find the products by 7, let him set in one column the figure 1 seven times, in another the figure 2 seven times, in a third the figure 3 as often, &c. Then the sums of these columns will be the products by 7.

It will also assist him if he attend to the relations subsisting between some of the successive products in the table. Thus, he will see that the products by 10 are formed simply by the addition of a cipher: that the first nine products by 11 are formed by repeating the figure: that the products by 5 terminate in 5 and 0, alternately; and that their first figures are each half of the number to be repeated, or of the one next below it: that in the successive products by 9, the first figure generally increases, and the second decreases, by a unit, &c.

Rule I. *When the multiplier does not exceed* 12: By means of the multiplication table, multiply each figure of the multiplicand by the multiplier, beginning with the units, and setting down and carrying as in addition.

Exam. 1. Multiply 5764 by 4.

Multiplicand.	5764
Multiplier ...	4
Product	23056

Set the numbers as in the margin, and proceed thus:—4 times 4 are 16; set down 6, and carry 1: 4 times 6 are 24, and 1 are 25; set down 5, and carry 2: 4 times 7 are 28, and 2 are 30; set down 0, and carry 3: 4 times 5 are 20, and 3 are 23; set down 23: and the product is twenty-three thousand, and fifty-six

5764
5764
5764
5764
23056

The same product may also be obtained by writing down, as in the margin, 5764 four times, and adding the four lines together. This exemplifies the principle stated on page 23 that multiplication by a whole number may be regarded as a substitute for a particular case of addition—the case, namely, in which the numbers or quantities to be added together are all equal.

Exercises.

1. 144 × 3	4. 936 × 6	7. 3046 × 9
2. 795 × 4	5. 1599 × 7	8. 7308 × 11
3. 729 × 5	6. 125 × 8	9. 1729 × 12

Rule II. *When the multiplier is greater than* 12: (1.) Place the multiplier below the multiplicand, with

labour of committing a still more extended table would be scarcely compensated by the advantage resulting.

13 times	14 times	15 times	16 times	17 times	18 times	19 times
2 are 26	2 are 28	2 are 30	2 are 32	2 are 34	2 are 36	2 are 38
3 — 39	3 — 42	3 — 45	3 — 48	3 — 51	3 — 54	3 — 57
4 — 52	4 — 56	4 — 60	4 — 64	4 — 68	4 — 72	4 — 76
5 — 65	5 — 70	5 — 75	5 — 80	5 — 85	5 — 90	5 — 95
6 — 78	6 — 84	6 — 90	6 — 96	6 —102	6 —108	6 —114
7 — 91	7 — 98	7 —105	7 —112	7 —119	7 —126	7 - 133
8 —104	8 —112	8 —120	8 —128	8 —136	8 —144	8 —152
9 —117	9 —126	9 —135	9 —144	9 —153	9 —162	9 —171

units under units, tens under tens, &c. (2.) Find by Rule I. the products of the multiplicand by the several figures of the multiplier, successively, setting the right-hand figure of each product under that figure of the multiplier which produces it. (3.) Add these products together, and the sum will be the total product required.

Exam. 2. How many days are there in 39 years, each year consisting of 365 days?

Set the numbers as in the margin; multiply by 9, and set 5, the first figure of the product, under the 9; multiply then by 3, and set 5, the first figure, under the 3. Then, by adding these partial products, we get fourteen thousand, two hundred, and thirty-five, the number of days required.

Multiplicand...	365
Multiplier......	39
	3285
	1095
Product.......	14235

Rule III. *If*, in either of the foregoing cases, *the multiplicand, or the multiplier, or both, end in ciphers*, the figures before the final ciphers may be arranged and multiplied in the manner that has been pointed out; and to the result as many ciphers must be annexed, as are found at the end of the factors.

Ciphers in any other part of the *multiplier* are to be neglected as multipliers, though they must be inserted in their proper places, so as to make the value figures range properly.

Exam. 3. Multiply 320 by 2400.

The numbers being arranged as in the margin, multiply 32 by 24; to the product, 768, annex three ciphers; and the entire product is seven hundred and sixty-eight thousand.

Multiplicand...	320
Multiplier......	2400
	128
	64
Product......	768000

Exam. 4. Multiply 36407 by 40206.

Here the first figure of the first partial product, is set below the figure 6 of the multiplier; the first figure of the second partial product, below 2; and the first of the third, below 4; the ciphers in the multiplier being neglected, except as means for making the figures in the partial products range properly.

Multiplicand...	36407
Multiplier......	40206
	218442
	72814
	145628
Product..	1463779842

Reason of the Rules.

The reason of the process in the first case, is evident from the illustration that has been given of the operations in addition.

Thus, in the example in the margin we have first 7 times 7 equal to 49, or 40 and 9; the 9 is set down and the 40 reserved. Then 7 times 9 are 63, and 4 are 67, or 60 and 7. The 4 added to 63 is equivalent to the 40 reserved, since, by the nature of notation, 4 in the second place is equivalent to 40 in the first; 7 is then set down, and 60 reserved; for which, by the nature of notation, we are to carry 6 to the next column.

Exam. 5.

$$\begin{array}{r} 497 \\ 7 \\ \hline 3479 \end{array}$$

In illustrating the second case, it is necessary to consider that *the annexing of one cipher to a number expressed by the modern notation, makes its value* 10 *times what it was before; the annexing of two ciphers* 100 *times; of three* 1000 *times, &c.* This is evident from the principles of notation; since the annexing of a cipher removes each figure one place to the left, and thus increases its value tenfold; while, by the annexing of a second cipher, this latter value is increased tenfold, and, consequently, the original value one hundredfold; and so on. Thus, the figure 7, with one cipher annexed, expresses seventy, or ten times 7; with two ciphers, seven hundred, or 100 times 7, &c. In like manner, if to 24 we annex a cipher, we get 240, which is 10 times as great as 24; the values expressed by the 2 and 4 being each increased in a tenfold degree.

Hence it follows, that to multiply a number by 20—that is, twice 10—we are to double it and annex a cipher: to multiply by 300—that is, three times 100—we treble the number, and annex two ciphers: to multiply by 7000, we multiply by 7, and annex three ciphers: and so on.

From these principles, the operation in the margin will be understood. The multiplier is equivalent to 300, 40, and 6, taken together; and, therefore, the multiplicand is multiplied by 6, 40, and 300, successively, and the sum of the products taken. In multiplying by 40, the product by 4 is found, and a cipher annexed; and in multiplying by 300, two ciphers are annexed to the product by 3. In the second, or common form of the operation, these additional ciphers are omitted, but the omission is compensated by making the figures occupy the same places which they do, when the ciphers are annexed; and hence the reason of setting the figures in the manner prescribed in Rule II. is evident. When the pupil is learning the reason of the rule for multiplication, he will find it advantageous for some time, to annex the ciphers, in the mode above explained.

Exam. 6.

$$\begin{array}{rl} 6758, & \textit{multiplicand.} \\ 346, & \textit{multiplier.} \\ \hline 40548 & = 6758 \times 6 \\ 270320 & = 6758 \times 40 \\ 2027400 & = 6758 \times 300 \\ \hline 2338268 & = 6758 \times 346 \end{array}$$

$$\begin{array}{rl} 6758, & \textit{multiplicand.} \\ 346, & \textit{multiplier.} \\ \hline 40548 & \\ 27032 & \\ 20274\! & \\ \hline 2338268, & \textit{product.} \end{array}$$

Methods of Proof.

1. Let the multiplier be taken as multiplicand, and the multiplicand as multiplier; and if the result thus obtained agree with the result found before, the process is right.*

Thus, when 64 is multiplied by 45, the product is found to be 2880; and when 45 is multiplied by 64, the same result is obtained.

$$\begin{array}{r} 64 \\ 45 \\ \hline 320 \\ 256 \\ \hline 2880 \end{array} \qquad \begin{array}{r} 45 \\ 64 \\ \hline 180 \\ 270 \\ \hline 2880 \end{array}$$

2. Commencing at either end of the multiplicand, proceed to add together its digits, neglecting 9 when it occurs; and, whenever your sum attained amounts to 9 or more, subtract 9 from it, and use the remainder instead of it; and at the close reserve the final excess. Proceed in like manner with the multiplier and the product. Then multiply together the excesses found in the multiplicand and the multiplier; and, in the same manner as before, find the excess in this product. If this excess and the excess previously found in the product of the given factors be the same, the work is *generally* correct: *if they differ, it must be wrong.* The operation of subtracting 9 from the sum attained whenever that sum amounts to more than 9, is most easily effected by adding together the two digits of that sum, and taking as the requisite remainder the sum of these two digits, which will be the same as would have been obtained by subtracting 9.†

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* This method of proof depends on the principle already stated and exemplified (see page 23), that the product of two numbers is the same whichever of them is taken as multiplier.

† This mode of proof is very easy and convenient in practice. In some cases, indeed, operations which are incorrect may *appear* by it to be correct; an occurrence, however, which is very rare, unless it be either designedly effected, or arise from the misplacing of figures; and hence, for the purposes of the experienced arithmetician, it is perhaps preferable to any other method.

The principle on which this process depends is, that *if any number and the sum of its digits be each divided by 9, the remainders are in both cases the same.* Thus, 1000=999+1, where the remainder must be 1, since 999 is evidently divisible by 9, without remainder. Hence, 3000 is equal to 3 times 999, together with 3 times 1, or 3; and, since 3 times 999 is divisible by 9, it follows, that if 3000 be divided by 9, the remainder must be 3. In like manner it might be shown, that in dividing by 9, the remainder in 5000 would be 5; in 400, 4; in 20000, 2, &c.; being, in all such cases, the same as the value figure contained in the number. Hence, if the number 487 be proposed, it is equivalent to 400+80+7; which parts, if divided successively by 9, would leave, on the foregoing principle, the remainders 4, 8, 7; and therefore the remainder of the entire number will evidently be the same as the sum of these, diminished by the rejection of 9 as often as possible.

Now, suppose it were required to multiply 112 by 48, we have 112 equal to 108, that is, 12 times 9, together with the excess 4; and therefore the required product will be 48 times 108, together with 48 times 4. But, since 108 contains 9 without remainder, its product by 48 will also contain it without remainder; and therefore the excess in the final product will be the same as the excess in the product of 48 into 4. Again, 48 is equal to 45, that is, 5 times 9, together with the excess 3; and therefore 48 times 4 is the same as 45 times 4, together with 3 times 4; and, since 45 contains 9 without remainder, 45 times 4 will also contain it without remainder. Hence the excess of the entire product will be the same as that of the product of 4 and 3, the excesses of the factors, 112 and 48; and thus the reason of this method of proof is evident.

In the annexed example, 6 and 8 are 14, which exceeds 9 by 5 (as may be seen either by directly subtracting 9 from 14, or by the easier process of adding together the digits 1 and 4 of the 14, their sum being 5, the same as the excess 5 got by subtracting 9 from 14): then 5 and 5 are 10; and the excess of this attained sum above 9 is 1, or instead we may prefer to say the sum of its digits is 1; 1 and 7 are 8, which excess we set opposite to the multiplicand. In the multiplier, in like manner, 8 and 7 are 15, and the sum of the digits of this attained sum is 6, and this excess is set opposite to the multiplier. In the product, 6 and 1 are 7, and 5 are 12, and the sum of the digits of this 12 is 3; 3 and 3 are 6, and 1 are 7, and 5 are 12, which has for the sum of its digits 3, to be set opposite to the product of the original multiplication as its final excess. Then the product of the two excesses, 6 and 8, is 48, the sum of the digits in which is 12, which again has the sum of its digits 3; that is to say, its excess above 9 is 3, the same as the excess in the product of the original multiplication; and hence we judge the work very likely to be correct.

```
      68597......8
        897......6
     ------
     480179
    617373
   548776
  --------
   61531509......3
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Exercises.

10. 1816 × 10	12. 40376 × 4000	14. 7854 × 900
11. 40376 × 40	13. 819200 × 700	15. 4096 × 12000

16. If, on an estate, 1572 yards of new thorn-hedge are to be planted, and there are to be 9 thorns in each yard of length; how many young thorns will be required? *Answ.* 14148.

The same property belongs to the digit 3; but, in practice, it is better to use 9.

Another method of proof, which is nearly as easy, and is perhaps more certain than the preceding, depends on a property of the number 11 in the decimal notation. This property might be proved in a manner similar to that given above. It would be too tedious, however, to be introduced here; and indeed both are better and more easily given by means of algebra. The following is the method referred to: *Commencing at the units of the multiplicand, add together the digits in the odd places, rejecting* 11 *as often as possible, and reserve the result; proceed in the same manner with the digits in the even places, and from the former result, increased, if necessary, by* 11, *take this result, and place the excess opposite to the multiplicand. In a similar way find the excesses of the multiplier and the product; then multiply the excesses of the two factors together, and find, in the same manner, the excess of their product: if this excess, and the excess of the product of the factors, be the same, the work is generally correct; if they differ, it must be wrong.*

In the annexed example (the work of which, for brevity, is omitted), we say, 3 and 9 are 12, which exceeds 11 by 1; 1 and 8 are 9; then 6 and 4 are 10, which being taken from 20 (=9+11), the excess is 10. Then, in the multiplier, the sum of 5, 0, and 4, is 9, from which 1, the excess of 8+4 above 11, being taken, the excess is 8. In the product, 5 and 8 are 13; 2, the excess, and 9 are 11, which is rejected; 5 and 7 are 12, the excess of which is 1; then 5 and 3 are 8, and 5 are 13; 2 and 4 are 6, and 3 are 9; 9 from 12 (=1+11), and the excess is 3. The product of the first two excesses is 80, the excess of which (found by taking 8 from 11) is 3, the same as the excess of the product of the factors.

```
Multiplicand,     84963....10
  Multiplier,     44085.... 8
                  -----
     Product, 3745593855.... 3
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By applying both these modes of proof to the same operation an almost absolute certainty of its correctness would be obtained.

Exercises.	*Answers.*
17. 958 × 34	=32572
18. 7198 × 216	=1554768
19. 31416 × 175	=5497800
20. 8862 × 189	=1674918
21. 7071 × 556	=3931476
22. 93186 × 4455	=415143630
23. 40930 × 779	=31884470
24. 12345 × 686	=8468670
25. 46481 × 936	=43506216
26. 16734 × 708	=11847672
27. 7575 × 7575	=57380625
28. 8320900 × 1328	=11050155200
29. 17500 × 732	=12810000
30. 15607 × 3094	=48288058
31. 7422153 × 468	=3473567604
32. 9264397 × 9584	=88789980848
33. 4687319 × 1987	=9313702853
34. 204053 × 1617000	=329953701000
35. 9507340 × 7071	=67226401140
36. 39948123 × 6007	=239968374861
37. 73885246 × 6079	=449148410434
38. 57902468 × 5008	=289975559744
39. 57902468 × 5080	=294144537440
40. 57902468 × 5800	=335834314400
41. 12481632 × 1509	=18834782688
42. 79068025 × 1386	=109588282650
43. 92948789 × 7043	=654638320927
44. 58763718 × 6754	=396890151372
45. 73084163 × 7584	=554270292192
46. 144 × 144 × 144	=2985984
47. 3851 × 3851 × 3851	=57111104051
48. 79094451 × 764095	=60435674536845
49. 79548050 × 97280	=7738434304000

Exer. 50. Multiply fifty-six millions, seven thousand, eight hundred, and fifty-four, by eighty millions, six hundred thousand, nine hundred, and seventy-six. *Answ.* 4514287696065504.

51. Multiply eighty millions, seven thousand, six hundred, by eight millions, seven hundred, and sixty. *Answ.* 640121605776000.

52. Required the amount of seven hundred and nine millions, four hundred and sixty-five thousand, nine hundred, and eight, repeated eight hundred thousand, three hundred, and sixty-five times. *Answ.* 567831681456420.

53. Multiply eight hundred and seventy-seven millions, five hundred and ten thousand, eight hundred, and sixty-four, by five hundred and forty-five thousand, three hundred, and fifty-seven. *Answ.* 478556692258448.

54. How many yards of linen are in 759 pieces, each containing 25 yards? *Answ.* 18975.

55. Sound is known to travel at about the rate of 1130 feet per second; how many feet will it travel in 69 seconds? *Answ.* 77970.

56. In respect to a clock which strikes the hours in the ordinary way; giving 1 stroke at 1 o'clock, 2 strokes at 2 o'clock, and so on, up to 12 strokes at 12 o'clock; and then beginning with 1 stroke at the next hour, 1 o'clock: find how many strokes it gives in 24 hours, and how many in a year of 365 days. This exercise involves, as may readily be noticed, both *simple addition* and *simple multiplication*.* *Answ.* 156 in 24 hours, and 56940 in a year.

57. If a watch ticks 240 times per minute, how many ticks does it make in 3 years of 365 days each, if it never stops, but goes perfectly right, during that time? *Answ.* 378432000.

58. Rees's "Cyclopædia" consists of 39 volumes, each containing, at an average, 774 pages of two columns. In each column there are 67 lines, each containing, at an average, 10 words, and in those 10 words there are, at an average, 47 letters. Make an estimate of the number of pages, lines, words, and letters contained in the entire work. *Answ.* 30186 pages, 4044924 lines, 40449240 words, 190111428 letters.

59. The distance from the earth to the sun is found to be about 23,114 times the earth's radius (the distance from the centre to the surface) at the equator, and that radius is very nearly 3963 miles. Required the distance between the earth and sun, in miles, according to these given numbers. *Answ.* 91600782.

INTRODUCTORY EXPLANATIONS FOR DIVISION.

If we fold a piece of cord by placing its two ends together and drawing it out so as to have two lines of cord lying alongside of one another, we shall have *divided* the cord, or its length, into two equal parts; and if we fold again this doubled cord so as to have four lines lying alongside of one another, we shall have *divided* the cord into four equal parts; but neither of these operations of division has been a case of division in arithmetic, as the length of the cord was not given numerically, and the result has not been brought out numerically. If, however, we are told that a piece of cord is 20 inches long, and if, as a result from reasoning, we find that the length of each of the four equal parts into which it might be divided would be 5 inches, we perform an operation in arithmetical DIVISION. In such cases as this, the meaning of the name *division*, as applied to them, is very obvious. In some other cases included

* In like manner as in the exercise here given, and in accordance with an intimation in the Preface, often in what follows in this treatise exercises given on any particular subject, or under any particular rule, may involve for part of their work the application of previously taught rules or methods.

under the same name, the connexion of the name with the operation is not so readily noticed; but a little consideration may show them all to be only varieties of one kind of operation.

Division, then, in arithmetic, has several somewhat distinct objects in different cases, the chief of which will now be shown; and, for simplicity, some cases will first be mentioned in which neither *remainders* nor *fractions* come into consideration.

(1.) To find how often one given number is contained in another. Thus, if it be wanted to find how often 3 is contained in 15, we shall find that it is contained 5 times; and the question is one in division. This operation may be looked on as equivalent to dividing 15 into a set of lots each containing 3, and *finding how many of these lots there would be.**

(2.) To find a number which will be contained a given number of times in another given number. Thus, if it be wanted to find the number which will be contained 3 times in 15, the answer will be that 5 is the required number; and the operation is one in division; and it may be regarded as being equivalent to dividing 15 into 3 equal lots, and *finding that each lot will consist of* 5. Another way of expressing the question in

* The operations and principles of arithmetic explained or mentioned here in the text, and many others of like kinds, may be brought very clearly before the mind by taking a sufficient number of any convenient objects, such, for instance, as pebbles or shells, and actually performing with them the operations indicated, using them as counters. For instance, to exhibit an operation of division belonging to the first case here mentioned in the text—namely, to find how often 3 is contained in 15:—place on a table 15 pebbles in a heap: then take out of this heap one lot of 3 pebbles, and then another lot of 3, and proceed taking out successive lots of 3 each, and laying them separate, until there be none remaining in the original heap. Then count how many of these lots there are, or, in other words, how often 3 is contained in 15. From the frequent use of pebbles among the ancients, as counters for the performance of arithmetical operations, the words *calculate* and *calculation* have come down to modern times, derived from the Latin word *calculus*, a pebble. It may readily be supposed that any such numerical notation as that of the ancient Romans, shown at page 14 of this book, would be very ill suited for affording easy means of doing arithmetical work by written characters alone; but it appears that they had in familiar use methods systematically contrived for bringing out the results in arithmetical questions by counters, which, whether real pebbles, as they sometimes were, or small pieces of ivory, wood, or other substance, were generally called by the name *calculus*. Such systems for the practical performance of arithmetic by counters continued in very frequent use in England and various other countries of Europe until the middle or end of the sixteenth century, and even in occasional use later still, and were elaborately taught in treatises on arithmetic in those times. Many who could neither read nor write could readily work the arithmetic requisite for their business affairs by the counters. In Shakespere's *Winter's Tale*, Act IV. scene 2; Clown talking of a sheep-shearing says, "Every tod yields pound and odd shilling: fifteen hundred shorn,—What comes the wool to?—I cannot do't without counters."

this case is to say that it is required to find the third part of 15, and the answer will be that each lot of 5 is the third part of 15.

The given number which is to be divided into lots is called the DIVIDEND; and the other given number which in one case is to be the number in each lot, and in the other case is to be the number of lots, is called the DIVISOR. The required number is usually called the QUOTIENT, a name which, according to its Latin derivation (from *quoties*, how often), would apply originally and properly to the answer to the question, *How often is one given number or quantity contained in another given number or quantity?* but in ordinary usage its meaning is often extended to apply to the answer in both the cases of division already treated of; and also to the answer in other cases following, when that comes out *without a remainder* or *without a fractional part*: but when there is a remainder or a fractional part in the answer, the word *quotient* is subject to an ambiguity of meaning; being applied often in two different senses in established usage; sometimes being made to include and sometimes not to include the fractional part. This will be more fully explained a little farther on.

It may now be noticed that in one of the two cases already treated of, the given divisor expresses how many there are to be in each lot, and the question is, *How many lots must there then be?* In the other case the divisor expresses how many lots there are to be, and the question is, *How many must there be in each lot?*

The foregoing two cases are concerned with abstract numbers, or numbers detached from the consideration of any one particular kind of thing which they count, but left applicable to things in general. The next two cases which will be mentioned are concerned with quantities of some particular thing expressed by numbers of units of that thing.

(3.) To find numerically a quantity which will be contained a given number of times in another given quantity numerically expressed. Thus, if the question is proposed :—to find the quantity of money which will be contained three times in £1 - 2 - 6, the answer, when found, will be 7*s*. 6*d*.; and the operation may be viewed as dividing £1 - 2 - 6 into three equal parts and finding

that each of those parts will amount to 7*s.* 6*d.* The answer here is a quantity of money.

(4.) To find the number of times that a given quantity is contained in another given quantity. Thus, if the question be proposed:—to find how often the quantity of money 10*s.* 6*d.* is contained in £1 - 11 - 6, the answer will be *that it is contained three times.* The answer here is simply a number of times.*

A few cases will next be briefly explained in which remainders or fractions arise to be considered in the answers.

(5.) To find how often a given number or quantity is contained in, or may be taken out of, another given number or quantity, and how much will remain over after it has been taken out as many times as it can be. Thus, it may be required to find how many lots of 3 pence each can be taken out of a heap consisting of 17 pence, and how much will remain over, being less than another lot of 3 pence. The answer, when found, will be that, out of 17 pence, 3 pence can be taken 5 times, and that there will then remain 2 pence, which is too small to form another complete lot of 3 pence. In the answer in this case the 5, or 5 *times*, is called the *quotient*, and the 2 pence is called the *remainder.*

(6.) Nearly the same question for solution is frequently put in a modified way which brings out an important distinction in the answer. Thus, it may be required to find how often a given number or quantity is contained in another given number or quantity, when a remainder would occur if the answer were to be given as in the last foregoing example; but it may be wanted that the answer shall be stated as *a fractional number of times*, instead of being stated by *a whole number of times and a remainder.* For instance, if the question is, How often can 3 pence be taken out of 17 pence? the answer is frequently given by a fractional number in saying that 3 pence can be taken out of 17 pence $5\frac{2}{3}$ times. But there is a manifest incongruity of ideas in speaking of taking the full 3 pence 5 times out of 17

* It was pointed out in a note on page 23 that it would be absurd to speak of multiplying money by money; but, from what is stated in the text here, it will be seen that there is no absurdity in dividing money by money, which means finding how often one sum is contained in another.

pence, and then taking the full 3 pence a fraction of another time out of it: and the form of answer sanctioned by usage, and often affording advantages of convenience and brevity of expression, that it can be taken $5\frac{2}{3}$ times out of 17 pence, must be interpreted to mean really that the full 3 pence can be taken out 5 times, and that, besides, $\frac{2}{3}$ of the same quantity of money can be taken out once. In this case the entire answer, $5\frac{2}{3}$, or $5\frac{2}{3}$ times, is called the *quotient*; and, when used in this way, the word quotient has a different meaning from that stated in the case immediately preceding, where only the number 5 was called the *quotient.*

The subject of fractions will be more fully discussed and explained farther on; but as fractional results continually arise in division, some elementary consideration or treatment of them forces itself inevitably both on the teacher and the pupil, even in the early stages of the pupil's progress.

The sign $\div$, called the *sign of division*, with one number placed before it and another after it, denotes that the number which precedes it is divided by the one which follows it. The division of one number by another is also frequently denoted by writing the dividend above the divisor, with a line drawn between them. Thus either of the short statements $144 \div 9 = 16$, or $\frac{144}{9} = 16$, denotes that 144 is divided by 9, and that the quotient is 16. Division in cases of other numerical expressions than mere numbers may be denoted in like manner. Thus we might write £16 - 12 - 8 $\div$ 4 = £4 - 3 - 2, or, $\frac{\pounds 16 - 12 - 8}{4} = \pounds 4 - 3 - 2$; either of which two statements would signify that if the sum of money £16 - 12 - 8 be divided into 4 equal parts, one of those parts will be £4 - 3 - 2. Also we might write £6 - 4 - 3 $\div$ £2 - 1 - 5 = 3, or, $\frac{\pounds 6 - 4 - 3}{\pounds 2 - 1 - 5} = 3$; either of which would denote that £6 - 4 - 3 is divided by £2 - 1 - 5, and that the quotient is 3, meaning 3 *times.*

A process of division is called SIMPLE DIVISION in the following cases,* provided that neither the divisor nor the dividend be fractional:—

* It might at first sight appear to some readers that the three cases noted here are not all really distinct, and, as a reason, it might be said that to divide 15 quarts of milk into 3 equal quantities is the same as to divide the number 15 into 3 equal parts. Let us, however, take another case, and an important distinction will become evident between the dividing of numbers and the dividing of quantities by aid of numbers of units.

A man and two boys go out in a boat to fish. They catch 39 fishes of various kinds, some large and some small, some much esteemed and some considered as almost worthless. They arrange among themselves that each person is to get

1st. When the dividend is simply a number, as, for instance, 12, 30, or 100, in which case it necessarily follows that the divisor must also be simply a number.

2nd. When the dividend is a quantity expressed by a number of units *all of one magnitude*, or, as it may be said, *all of one denomination*, and the divisor is simply a number.

3rd. When the dividend is a quantity expressed in units of one denomination, and the divisor another quantity of the same kind expressed in units of the same denomination.

It is to be observed that although these three cases of simple division are distinct in principle, yet the arithmetical process is identically the same in all.

The name COMPOUND DIVISION is usually taken to distinguish those cases in which the dividend is a quantity compoundly expressed in units of different denominations, and the divisor a mere number. But more generally DIVISION is properly to be called COMPOUND when, either in the dividend or in the divisor, or in both jointly, different denominations, or, as it may be said, unequal units, anywhere occur.

Thus, if £1 - 7 - 6 is to be divided by the number 3, or by 2*s.* 6*d.*, or if £25 is to be divided by £8 - 6 - 8, the process may be called COMPOUND DIVISION in each case.

SIMPLE DIVISION.

THE nature and chief objects of DIVISION in general, and the distinction between SIMPLE and COMPOUND DIVISION, having been explained in the foregoing chapter, rules and explanations for SIMPLE DIVISION will now be given.

RULE I. *When the divisor does not exceed* 12: (1.)

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the same number of fishes as each of the rest, but that the man is to get chiefly the larger fishes, and that the boys are to get chiefly the smaller ones. The entire lot of fishes caught is then divided numerically into 3 lots equal to one another in number—the entire number, 39, being divided into 3 equal numbers, 13 each—but this is not at all a case of dividing a quantity of fish (or of anything whatever) into three equal quantities.

It is a very different kind of operation from dividing a quantity of milk, 39 quarts, into 3 equal quantities, each found as being 13 quarts.

The two operations may, no doubt, be performed by use of one and the same arithmetical process; but the practical results sought for and attained in the two cases are very distinct. In the one case there exists no equality of quantity in the lots arrived at, while in the other case the resulting lots are perfectly equal to one another in quantity.

Place the divisor before the dividend with a line between them. (2.) By the multiplication table, find how often the divisor is contained in the first figure, or, if necessary, in the number expressed by the first two, or the first three figures of the dividend; and write down for the commencement of the quotient the figure denoting the number of times. (3.) Find the product of this figure and the divisor, and take it from the number expressed by the figure or figures of the dividend formerly used. (4.) To the remainder annex, mentally, the next figure of the dividend; and proceed as before, to find the next figure of the quotient, and the next remainder; and continue the process till all the figures of the dividend have been employed. (5.) If there be a remainder at the conclusion, note it as a remainder after the quotient previously found; or else write it, with the divisor under it, as a fractional addition to the previous quotient, in order to find the quotient under another of the ordinary significations of the term quotient.*

Method of Proof.—Find the product of the divisor and quotient, and add to it the remainder; if the sum be equal to the dividend, the work is correct.†

Exam. 1. Let it be required to find how many weeks there are in 1738 days.

To solve this question, since a week consists of 7 days, we have merely to find how often 7 is contained in 1738: that is, to divide 1738 by 7. Setting then the divisor and dividend as in the margin, we say, 7 is contained in 17 (or, for brevity, 7 in 17) twice and 3 over, and we write 2, for the first figure of the quotient, under the 7 of the dividend: then the remainder 3 and the next figure 3 annexed mentally to it, express 33;—7 in 33, four times and 5 over, and we write 4 under the 3: then to the remainder 5 annexing, mentally, the 8, we have 58;—lastly, 7 in 58, eight times and 2 over, and we set 8 under the 8, and next we write the remainder 2 with the divisor 7 under it, to complete the quotient. It thus appears, that in 1738 days there

$$\begin{array}{r} 7)\underline{1738} \\ 248\tfrac{2}{7}, \textit{ quotient.} \\ \underline{7} \\ 1738, \textit{ proof.} \end{array}$$

* The operation according to this rule is generally termed *short division*, while that according to Rule II. is called *long division*.

† The work may also be proved in either of the following methods:—

1. Subtract the remainder from the dividend, and divide the result by the quotient, taken as the whole number without fractional part; if the quotient thus found be the same as the given divisor, the work is right.

2. *Cast the nines* out of the divisor, dividend, quotient, and remainder; then to the product of the excesses of the divisor and quotient add the excess of the remainder, and cast the nines out of the sum; if this excess be equal to the excess of the dividend, the work may be presumed to be correct.

are 248 complete weeks, and two days, or two sevenths of another week.

Exam. 2. If 10,507 pounds sterling were divided equally among 12 persons, what would be the share of each?

The object in this question is to find a number of pounds which will be contained 12 times in the given number of pounds 10,507, or which, as it is termed, will be one twelfth part of 10,507 pounds. To do this in the usual way, we proceed by finding how often 12 is contained in 10,507. Then in performing the operation, 12 is not contained in the first figure 1, nor even in 10.

$$\begin{array}{r} 12)\underline{10507} \\ 875\tfrac{7}{12},\ \textit{quotient.} \\ \underline{12} \\ 10507,\ \textit{proof.} \end{array}$$

Taking, therefore, the first three figures, we say, 12 in 105, eight times, and 9 over; 12 in 90, seven times, and 6 over; 12 in 67, five times, and 7 over; and the 7, with the divisor 12 below it, is written to complete the quotient. Then the share of each would be 875 pounds, with a twelfth part of the 7 pounds that remain at the end of the operation, as will next be explained.

The operation here performed may be regarded in more ways than one, and the following is a simple way:—Every time that a sum of 12 pounds can be taken out of 10,507 pounds, one pound can be given to each of the 12 persons. Hence when it is found that 12 pounds can be taken out of 10,507 pounds 875 times, leaving a remainder of 7 pounds, it follows that each of the persons can obtain one pound 875 times, and that there will remain 7 pounds still to be divided among the 12 persons, and that then, out of this remainder, each person will get one twelfth of 7 pounds. The whole quantity which each will receive is then expressed as $875\frac{7}{12}$ pounds, which may be read in various ways. Thus it may be read as 875 pounds and a twelfth of 7 pounds, or as 875 pounds and seven twelfths of another pound.

As 7 pounds are seven times as great as 1 pound, a twelfth part of 7 pounds is evidently 7 times as great as a twelfth part of 1 pound. It is plain, therefore, that the expression $\frac{7}{12}$, regarded at first as denoting a twelfth of 7 pounds, is the same in value as seven twelfths of 1 pound; and hence it is usually read *seven twelfths* of a pound. In like manner, $\frac{5}{8}$ is read *five eighths*; $\frac{1}{6}$, *one sixth*; $\frac{4}{11}$, *four elevenths*, &c. It would be shown also, in a similar manner, that, if a shilling be the unit, the fraction, $\frac{2}{5}$, means either two fifths of one shilling, or one fifth of two shillings; while, in reference to a ton, $\frac{3}{8}$ means equally three eighths of one ton, or one eighth of three tons.

In any fraction, expressed in the manner that has been now explained, the upper number is called its NUMERATOR, and the lower its DENOMINATOR. Thus, in $\frac{2}{3}$, 2 is the numerator, and 3 the denominator.*

* The term *numerator* means *numberer*, and *denominator* means *namer*. Thus in the fractions $\frac{2}{7}$, $\frac{3}{7}$, $\frac{4}{7}$, the *names*, or *denominations*, are sevenths; while the numerators, 2, 3, 4, show the *numbers* of those sevenths expressed by the several fractions.

Exercises.

1. 470850 ÷ 3	4. 3782047 ÷ 6	7. 74593822 ÷ 9
2. 1829765 ÷ 4	5. 7165537 ÷ 7	8. 53248675 ÷ 11
3. 4265983 ÷ 5	6. 27459332 ÷ 8	9. 49275189 ÷ 12

RULE II. *When the divisor exceeds* 12: (1.) Place the divisor to the left of the dividend, with a line between them, and leave a space to the right of the dividend for containing the quotient. (2.) Separate off mentally from the left hand of the dividend the least number of its figures which will make a number not less than the divisor. Consider the number formed by the figures so separated as a *subordinate dividend*; and find how often the divisor is contained in it,* and set this number of times as the first figure in the quotient. (3.) Multiply the divisor by the figure thus found, and set the product below the number designated as the subordinate dividend, and subtract it from that number. (4.) To the remainder annex the next figure † of the dividend, and consider the number so obtained as a new subordinate dividend now to be used. Then if this subordinate dividend be not less than the divisor, find how often the divisor is contained in it, and put this number of times as the next figure in the quotient to the right of what is there already; but if the subordinate dividend now arrived at be less than the divisor, put a cipher as the next figure in the quotient, and make another new subordinate dividend by annexing to the one last obtained the next following figure of the dividend; and, if necessary, go on repeating the process of putting a cipher in the quotient and annexing a figure of the dividend to the last subordinate dividend, till a new subordinate dividend be obtained which is not less than the divisor; then find how often the divisor is contained in the number so obtained, and put this number of times as the next figure in the quotient. Multiply in either case the divisor by the last figure obtained for the quotient, and subtract the product from the last subordinate dividend. (5.) Using the new remainder thus found, proceed again as in (4.); and go on until all the figures of the

* Instructions for aiding in the finding how often the divisor is contained in this subordinate dividend, and in the others which successively follow, are given in connexion with the present rule at its end.

† It is proper, for preventing mistakes, to put a dot below each figure of the given dividend when it is *brought down*.

dividend are exhausted.* (6.) If there be any remainder at the end, deal with it as directed in Rule I.

The finding how often the divisor is contained in each of the successive subordinate dividends, may be facilitated by first neglecting all the figures of the divisor except the first one or two at the left, and neglecting as many corresponding figures from the subordinate dividend, and then using the numbers remaining, instead of the numbers themselves, for a trial to find the number to put in the quotient. Then, by actually multiplying the divisor by the trial number so obtained, it can readily be seen whether that number is too great or too small, or the right one. It is too great if the product is greater than the number from which the product should be subtracted; and it is too small if the remainder left after the subtraction comes to be equal to, or larger than, the divisor. By practice, facility may be attained for arriving speedily at the proper figure to place in the quotient; and modes of assisting the judgment in guessing nearly the proper figure will be readily noticed without their being formally set down here as rules.

Exam. 3. Divide 15967 by 57.

```
57)15967(280 7/57        280 7/57
   114                    57
   ---                  ----
    456                 1967
    456                1400
    ---                -----
      7                15967, proof.
```

Let the divisor, 57, be set before the dividend, 15967, as in the margin. Observe that the figures 159 will form the first subordinate dividend, and proceed thus:—How often is 5 contained in 15? twice; † place 2 in the quotient, multiply the divisor by it, and set the product below the subordinate dividend 159. This being subtracted from 159, the remainder is 45, to which 6, the next figure of the dividend, is annexed. Again, how often 5 in 45? 8 times: place 8 in the quotient, proceed as before, and there is no remainder. Then 7, the remaining figure of the dividend, containing 57 no times, a cipher is placed in the quotient, and the remainder is written in the quotient over the divisor 57. The quotient, therefore, is $280\frac{7}{57}$.‡

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* It is carefully to be observed that for the set of figures at first separated mentally from the left of the dividend, and for every subsequent figure of the dividend brought down, a figure must be put in the quotient: and hence it is easy to know, even at the beginning of the work, how many figures the quotient must contain. Attention to this may often prevent great mistakes, more especially in cases when ciphers have to be put in the quotient, as errors of inadvertence are rather liable to occur in respect to the number of ciphers inserted.

† It would seem here, at first sight, that 3, and not 2, should be put in the quotient. Were the divisor multiplied by 3, however, the product, 171, would be greater than 159. When the second figure of the divisor is above 5, in *trying* for the figure to be placed in the quotient, the first may be increased by unity. Thus, in the example before us, we might have said, how often 6 in 15? &c.

‡ The French place the divisor to the right of the dividend, and the quotient below it, as in the example in the margin. This mode gives the work a more compact and neat appearance, and possesses the advantage of having the figures of the quotient near the divisor, by which means the practical difficulty of multiplying the divisor by a figure placed at a distance from it is removed. This difficulty every one must have felt, particularly in long operations; and hence this arrangement might, with propriety, be adopted in preference to that which is employed in this country.

```
59122 { 82, divisor.
574   { ---
---   { 721, quotient.
 172
 164
 ---
   82
   82
   --
```

Rule III. *If the divisor ends in one or more ciphers,* cut them off, to contract the operation, and cut off the same number of figures from the right of the dividend; then proceed with the remaining figures, according to one of the preceding rules, and to the last remainder annex the figures cut off in the dividend, to find the true remainder.

Exam 4. Divide 782967 by 3700.

In this exercise two ciphers are cut off from 3700, and two figures from the dividend; then the division by 37 proceeds in the common way, and to the last remainder, 22, the figures cut off are brought down, and the true remainder is 2267.

```
37,00)7829,67(211 2267/3700
      74            3700
      --            ----
       42           2267
       37         1477
       --         633
        59        ------
        37        782967, proof.
        ----
        2267, remainder.
```

Hence, when the divisor is unity, with one or more ciphers annexed, the quotient is found by cutting off from the dividend as many figures for remainder, as there are ciphers in the divisor.

Thus, if it be required to divide 53826 by 100, the quotient is simply $538\frac{26}{100}$.

Reason of the foregoing Rules.

The principle on which the operations in division depend, is, that *a part* of the quotient is found, and the product of it and the divisor deducted from the dividend; then *another part* of the quotient is found, and its product by the divisor taken from the remainder obtained before; and thus the operation proceeds till nothing remains, or till the remainder is less than the divisor. This will be understood from the following example:—

Exam. 5. Divide 31278 by 73.

Here, the first part of the quotient is 400, the product of which and 73, is 29200. This taken from the dividend, leaves 2078 still to be divided by 73. The next part of the quotient is 20; the product of which and 73, is 1460, which still leaves a remainder of 618 to be divided by 73. This gives 8, with the remainder 34. Hence, it appears that 73 is contained in the dividend 400 + 20 + 8 times, or 428 times, with the remainder 34. By comparing this, and the common process subjoined, it will be found that the latter is merely an abbreviation of this, the ciphers being omitted in the one and retained in the other.

```
73)31278(400
   29200
   -----
73)2078( 20
   1460
   ----
 73)618(  8
    584 ---
    --- 428
     34

73)31278(428 34/73
   292
   ---
    207
    146
    ---
     618
     584
     ---
      34
```

RULE IV. *When the divisor can be resolved into two factors, neither exceeding* 12, *and each being an integer, or whole number*, the dividend may be divided by one of them by short division, and the result by the other. In proceeding thus, if there be remainders, the true remainder on the whole division is found by multiplying the last remainder by the first divisor, and adding to the product the first remainder.

Exam. 6. Divide 20478 by 56.

Here the first remainder is 3, and the second 5, and by the rule we obtain 38 for the remainder in the division by 56.

$$\begin{array}{r} 7)\underline{20478} \\ 8)\underline{2925}\ldots3 \\ 365\ldots5 \\ 365\frac{38}{56} \end{array}$$

The special reason for this Rule IV. may be understood through the following considerations:—

2925 is 8 times 365 with 5 added,
and 20478 is 7 times 2925 with 3 added,
or is 7 times (8 times 365 + 5) with 3 added; *
or is 56 times 365 + 35 + 3;
or is 56 times 365 + 38.

Or the reason may be understood otherwise through the following considerations, in the statement of which the word quotient is to be taken in its sense of signifying the integer or whole number, and the remainder is spoken of as a distinct number.

(1.) For every unit in the first quotient, 2925, obtained by the division by 7, there are 7 units in the given dividend; and it has 3 units besides, as is shown by the remainder 3.

(2.) Again, for every unit in the second quotient, 365, obtained by the division by 8, there are 8 units in the first quotient, and that first quotient has 5 units besides, as is shown by the remainder 5.

(3.) Then for every unit in the second quotient, 365, there are 7 times 8 units, or 56 units, in the given dividend, and something more besides; this additional number being, as may be seen from what was said in (1.), 7 times the 5 units of the first quotient which remained over in the second division; and the original dividend contains 3 units more besides.

(4.) That is, the original dividend contains exactly the following, viz.:—

56 times 365, and 7 times the remainder 5, and once the remainder 3.

Introductory remark for Rule V.—It is often the case that a number is required to be divided by the product of three or more given factors; and it may also some-

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* In this it is to be understood that the double mark made thus ( ), and which is called a *bracket* or *vinculum*, has the effect of connecting all that it encloses, so that all shall be affected by the multiplier outside of the vinculum; that is to say, shall be taken 7 times.

times be convenient to resolve a given divisor into three or more factors. In such cases the following rule may often be used with advantage.

RULE V. *If the divisor is given as, or is resolved into, more than two factors, none of them exceeding* 12, *and each being an integer*, the dividend may be divided by one of them by short division, and the result by another, and so on till all of them have been used; and the true remainder, if there be any remainder, will be found by successive applications of the method stated in *Rule IV.* thus:—(*a.*) Confine attention to the last pair of remainders and the divisor which gave the earlier of them: call that divisor, for the present, the *initial divisor*; and multiply the last of the pair of remainders by this initial divisor, and add to the product the first of these remainders. (*b.*) Now use the result just found, taking it as the last remainder in a new pair of remainders now to be dealt with, and taking as the first in this pair the remainder next before those already used; call now the divisor which gave this first remainder the *initial divisor*. Then multiply the last of the pair of remainders by the initial divisor, and add to the product the first of these remainders. (*c.*) The result just found will be the required remainder, if the divisor was resolved into three factors; or, what is the same, if all the remainders have now been used: but if the number of factors has been more than three, proceed again according to (*b.*), and so go on till all the remainders have been used. Then the result finally obtained will be the required remainder.

Remark in reference to Rules IV. and V.—It is to be observed that if any of the factors leaves no remainder, a remainder zero is to be counted as being left by it for the due application of Rules IV. and V.

Exam. 7. Divide 53217 by $8 \times 6 \times 7$.

$$\begin{array}{r}8\underline{)53217}\\6\underline{)6652...1}\\7\underline{)1108...4}\\158...2\end{array}$$

Here, under *Rule V.*, the last pair of remainders are 4 and 2, and the initial divisor to be used with that pair is 6: and we multiply the remainder 2 in this pair by the initial divisor 6, and obtain the product 12, and to this we add 4, the first remainder of the pair, and obtain 16. Next, we use this result 16 as the last remainder of a new pair in which the first is 1, and for which the initial divisor is 8: and we multiply 16 by 8, finding 128, and to this product we add 1; and so obtain the required remainder 129. The quotient then may be

stated as 158 *times* with a remainder 129; or, in the other ordinary sense of the term quotient, it may be stated as $158\frac{129}{336}$; as the divisor $8 \times 6 \times 7$ is equal to 336.

It will be well for the pupil to work this example by taking the given factors in various orders; and also to find the result by long division, taking 336 as the divisor. The same result ought to be obtained in all cases.

*Exercises.**	*Answers.*	*Exercises.*	*Answers.*
10. $67359 \div 21$......	$=3207\frac{12}{21}$	15. $421645 \div 74$......	$=5697\frac{67}{74}$
11. $88763 \div 32$......	$=2773\frac{27}{32}$	16. $325000 \div 85$......	$=3823\frac{45}{85}$
12. $47839 \div 42$......	$=1139\frac{1}{42}$	17. $400000 \div 96$......	$=4166\frac{64}{96}$
13. $75043 \div 52$......	$=1443\frac{7}{52}$	18. $999999 \div 47$......	$=21276\frac{27}{47}$
14. $93840 \div 63$......	$=1489\frac{33}{63}$	19. $352417 \div 29$......	$=12152\frac{9}{29}$

Exercises.	*Answers.*
20. $51846734 \div 102$	$=508301\frac{32}{102}$
21. $727846489 \div 408$................	$=1782711\frac{401}{408}$
22. $980263711 \div 809$................	$=1211698\frac{29}{809}$
23. $536819237 \div 907$................	$=591862\frac{403}{907}$
24. $1457924651 \div 1204$............	$=1210900\frac{1051}{1204}$
25. $28101418481 \div 1107$	$=25385201\frac{874}{1107}$
26. $513513513513 \div 917$	$=559992926\frac{871}{917}$
27. $643751624367 \div 731$	$=880645176\frac{711}{731}$
28. $465465465465 \div 644$	$=722772461\frac{581}{644}$
29. $347382600435 \div 727$	$=477830261\frac{688}{727}$
30. $1700649160000 \div 759$.........	$=2240644479\frac{439}{759}$
31. $571824753344 \div 839$	$=681555129\frac{113}{839}$
32. $245379633477 \div 1263$.........	$=194283161\frac{1134}{1263}$

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* The pupil should prove the work of the several exercises in division by multiplication. Every operation in long division, thus proved, affords him an exercise in each of the four fundamental rules; as, in finding the quotient, he employs division and subtraction, and in the proof, multiplication and addition. Hence, perhaps the exercises here given will not seem, on due consideration, too numerous, though at first sight they might appear to be so. It is of the first importance that the pupil should have acquired both accuracy and despatch in performing the operations in the fundamental rules before he proceeds to apply them in the more advanced parts of arithmetic.

In proving the operation in this way, the mode which is generally best for adding in the remainder, is to write it above the first of the partial products. Thus, the proof of exercise 23 will be as in the margin.

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   591862
      907
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      403
  4143034
 5326758
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536819237
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When the pupil has had some practice in the methods of division already explained, he may omit writing the products in performing long division, which will at least save room in his operations. This method will be understood from the following example:—

Here the first figure put in the quotient is 2; then we say, twice 3 are 6; 6 from 6, and nothing remains; twice 7 are 14; 14 from 23, and 9 remain; twice 1 are 2, and 2 (carried) are 4; 4 from 4, and nothing remains. We next bring down 9, and place 5 in the quotient; then 5 times 3 are 15; 15 from 19, and 4 remain; 5 times 7 are 35, and 1 (carried) are 36; 36 from 40, and 4 remain; 5 times 1 are 5, and 4 (carried) are 9; 9 from 9, and nothing remains, &c.

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173)436951(2525 126/173
     909
      445
       991
       126
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	Exercises.	*Answers.*
33.	5368475555555 ÷ 1138.........	= 4717465333$\frac{1021}{1138}$
34.	234516447519 ÷ 759	= 308980826$\frac{585}{759}$
35.	1111111111111 ÷ 854.........	= 1301066874$\frac{715}{854}$
36.	789012345676 ÷ 7384.........	= 106854326$\frac{2492}{7384}$
37.	75843689426 ÷ 8593	= 8826211$\frac{8303}{8593}$
38.	65358547823 ÷ 2789	= 23434402$\frac{645}{2789}$
39.	33333333333 ÷ 5299	= 6290495$\frac{328}{5299}$
40.	321987653321 ÷ 7766.........	= 41461196$\frac{5185}{7766}$
41.	542713060315 ÷ 4444.........	= 122122650$\frac{3715}{4444}$
42.	2652104208416 ÷ 7539	= 351784614$\frac{3470}{7539}$
43.	7314617334753 ÷ 6784	= 1078215998$\frac{4321}{6784}$
44.	3146173847837 ÷ 9387	= 335162868$\frac{5921}{9387}$
45.	555777999444777 ÷ 891......	= 623768798478$\frac{879}{891}$
46.	765809034537648 ÷ 793......	= 965711266756$\frac{140}{793}$
47.	582390171945110 ÷ 693......	= 840389858506$\frac{452}{693}$
48.	582390171945110 ÷ 593......	= 982108215759$\frac{23}{593}$
49.	582390171945110 ÷ 493......	= 1181318807190$\frac{440}{493}$
50.	582390171945110 ÷ 393......	= 1481908834465$\frac{365}{393}$
51.	582390171945110 ÷ 293......	= 1987679767730$\frac{220}{293}$
52.	582390171945110 ÷ 193......	= 3017565657746$\frac{132}{193}$
53.	3333333333333333 ÷ 483 ...	= 6901311249137$\frac{162}{483}$
54.	3333333333333333 ÷ 484 ...	= 6887052341597$\frac{385}{484}$
55.	1000000000000000 ÷ 81......	= 12345679012345$\frac{55}{81}$
56.	1000000000000000 ÷ 729 ...	= 1371742112482$\frac{622}{729}$
57.	1000000000000000 ÷ 111 ...	= 9009009009009$\frac{1}{111}$
58.	1000000000000000 ÷ 1111...	= 900090009000$\frac{1000}{1111}$
59.	1000000000000000 ÷ 11111	= 90000900009$\frac{1}{11111}$
60.	555555555555 ÷ 123456......	= 4500028$\frac{98787}{123456}$
61.	555555555555 ÷ 654321......	= 849056$\frac{384579}{654321}$
62.	102030405060 ÷ 123456......	= 826451$\frac{70404}{123456}$
63.	908070605040 ÷ 654321......	= 1387805$\frac{649635}{654321}$
64.	3784926474826 ÷ 384365 ...	= 9847219$\frac{143891}{384365}$
65.	467817938473 ÷ 2100.........	= 222770446$\frac{1873}{2100}$
66.	367817938429 ÷ 36500	= 10077203$\frac{28929}{36500}$
67.	267817938473 ÷ 8760.........	= 30572824$\frac{233}{8760}$
68.	167817938176 ÷ 957000......	= 175358$\frac{332176}{957000}$
	167817938473 ÷ 87700	= 1913545$\frac{41973}{87700}$
	1360000 ...	= 196924$\frac{1298115}{1360000}$
	7000	= 4227792$\frac{34473}{87000}$

ıtain in the neighbourhood of Belfast, Blanc 15,680 feet: how many moun- ıer, must be piled one above another; r? *Answ.* 10$\frac{260}{1542}$.

73. If it be supposed, as in common circumstances is found to be nearly true, that as many persons die in 33 years as are equal to the entire population, it is required to find how many persons die annually, at an average, out of every million? *Answ.* 30,303, nearly.

74. How many lessons of ninety-five lines each, are contained in Virgil's Æneid, the number of lines contained in that poem being nine thousand, eight hundred, and ninety-two? *Answ.* $104\frac{12}{95}$.

75. The earth's equatorial and polar diameters are 41,847,426 and 41,707,620 feet respectively: divide each of them by their difference. *Answ.* 299 and 298; *rem. in each case* 45432 feet.

FRACTIONS:—INTRODUCTORY CHAPTER.

In the chapter of Introductory Explanations at the commencement of this treatise, some preliminary information has been given in respect to the nature of *fractional numbers*, or *fractional numerical expressions*, and their relation to *whole numbers*, or to *numbers* properly so called. Also in the explanations on division, the subject of fractions has been touched on in more than one place, as the consideration of the remainders left in division naturally and unavoidably brings fractions into notice for practical expression and treatment. The exposition of the subject of fractions hitherto introduced has, however, been very brief, and has consisted chiefly in some fundamental explanations of the nature of fractions in arithmetic, and of their relations to proper or "whole" numbers; and in some explanations of nomenclature used in reference to them alone, or to them conjointly with proper numbers. In respect to the nature of fractions in arithmetic some notions have been offered for consideration as to fractional things, such as parts of the indefinitely divisible units of quantity (yards, feet, tons, gallons, hours, &c.) used for affording numerical expression for quantities of things, or for answering the question *how much* (as when we speak of three fifths of a gallon, or of four tenths of a ton); and also notions have been offered of fractions of groups of single indivisible objects, the groups and fractions of groups being used for affording a certain kind of numerical expression—a fractional expression—different from an ordinary number, in reply to the question *how many* of those single objects there are (as when we speak of $2\frac{1}{2}$ dozen, or $\frac{3}{4}$ of a million). In the present chapter some of the most simple and essential principles of fractions, and methods of managing them, will be taught, especially those which are most useful to a learner of arithmetic at an early stage of his progress, while the more intricate processes, which can well be dispensed with at first, will be reserved for more advanced chapters in this book.

If a piece of tape one yard long is divided equally among eight persons, each person is said to get one eighth of a yard; and this share for each is called a *fraction* of a yard. If a piece of tape five yards long is divided equally among eight persons, each person is said to get one eighth of the five yards; and this share also is called

a fraction of a yard, the yard being regarded as the unit which is broken up into fractions.* In the second case it is obvious that the length which each person gets is five times as much as each got in the first case. This may be seen by considering that the length to be divided was five times as much in the second case as in the first, while the number of persons among whom the division was made was the same in both. Or otherwise, the same may be seen by conceiving, in the second case, every yard of the five to be marked out in eight equal parts, one for each person, so that there would be for each person, on the whole, five of those equal parts; or each person would get five *eighths of a yard.* Thus we see that *an eighth of five yards* is the same length as *five eighths of one yard.* The yard here is the unit in which lengths are counted, and the length which each person gets as his share is called a fraction of this unit. The explanations given in the foregoing example will aid the comprehension of the more general statements following.

If any *unit of quantity* (as, for instance, a yard, an inch, a pint, or an hour) be divided into any number of equal parts; and if one or more of those parts be taken, the quantity so obtained is called a *fraction of that unit*, or simply it is called a *fraction*, in relation to that *unit*, or that *quantity called one.*†

Likewise in the case of expressing *numbers of objects* by taking the objects in groups (as, for instance, in millions, or dozens, or in groups of any other number each; it may be in twenty-fives, or in twenties, or in thirteens, &c., when convenient for any reason), if any group or number of objects treated as a *unit*,‡ or considered as *one* of something, *one* group, or *one* whole, be divided into any number of equal parts into which it may be divisible; and if one or more

* It is to be observed, however, that we may also, if we please, perfectly well treat this eighth part of five yards as a fraction of the length five yards. It is the fraction called one eighth, and denoted $\frac{1}{8}$, of that length. In that mode of thought we may regard the length five yards as a certain unit, or single whole, of which the part under consideration, $\frac{1}{8}$th of it, is a fraction or broken portion. The two modes of thought, one stated above in the text and the other stated here in the foot note, are quite consistent mutually, and it is proper that both should be understood.

† In some considerations—often, for instance, in algebra—any portion of a unit may be called a fraction of the unit, whether admitting or not of being arrived at by dividing the unit into a number of parts all equal, and then taking an exact number of these parts. In arithmetic, however, only such quantities can be numerically expressed perfectly, and dealt with exactly, as admit of being arrived at in that way, and only such are called arithmetical fractions. By making the equal parts small enough and numerous enough, we can arrive at any requisite degree of exactitude in the expressing of a quantity which cannot be made up out of any exact number of exactly equal parts of a unit. For instance, it is known to mathematicians that if a square has its diagonal a unit in length, it is impossible to divide that unit into any number of equal parts such that an exact number of those parts will just make up a length equal to a side of the square; but by taking the equal parts small enough, we can have as small an error as we please in stating the length of the side as a certain number of those small equal parts of the diagonal. The side and diagonal of the square are called *incommensurable* with one another; and there are numberless other *incommensurable quantities* met with in mathematics.

‡ On this subject the reader may refer to remarks in the chapter of Introductory Explanations, page 4, as to a group or assemblage of single things being an object of our conception, which may, when we please, be treated as a *single thing*, or as a *unit.*

of those parts be taken, the partial group or number so obtained is called a *fraction* in relation to the whole group regarded as *one* whole, or as a *unit*, or *one*. Thus if we regard a group of twenty objects as a *unit*, *one* twenty (while we might have *two* twenties, or *three* twenties, or any *number* of twenties), then one fifth of that unit or of that one whole would be four; and three fifths of that unit or of that whole would be twelve; and so we say that twelve is a fraction of twenty, and we name that fraction as *three fifths*.

A fraction is ordinarily expressed by two numbers, or TERMS, called the *numerator* and the *denominator*.

The DENOMINATOR is written below the *numerator*, and expresses the number of equal parts into which the *unit* (whether a unit of quantity or a unit group of objects which are counted in number) is divided; and the NUMERATOR expresses the number of such parts denoted by the fraction.

Thus $\frac{4}{5}$, which is read *four fifths*, is a fraction, and signifies that a unit of some kind—a day, for instance—is divided into five equal parts, and that four of these parts are taken.

The word *numerator* means *numberer*, and *denominator* means *namer*. In the foregoing example $\frac{4}{5}$, the numerator 4 states the number of parts, and the denominator names them as *fifths*. It may here be noticed that when we choose to think upon the expression $\frac{4}{5}$ as *one fifth of four* (which we are quite free to do, as has been already explained in some previous passages; see pages 5 and 38, and foot note, page 47), the names numerator and denominator do not convey any etymological meaning in reference to that mode of thought, but they are then used as names without regard being taken to their derivation.

According to the explanations of fractions already given, under which any fraction is less than the *unit*, or *one thing*, or *one whole*, which is divided into parts, it follows obviously that in a fraction the numerator should be less than the denominator. It is often found, however, convenient in practice, and consistent with true modes of thought, to extend the meaning of the word fraction, so as to make it include such numerical expressions as $\frac{4}{3}$, $\frac{7}{5}$, or $\frac{8}{8}$. In respect to these, it may be noticed that eight eighths of any thing would be *one* of that thing; so that a fraction having its numerator equal to its denominator is equivalent to *one*, or denotes a unit of whatever is counted in number; which things counted may, for instance, be yards, days, gallons, dozens, millions, or various other divisible *units*. When the numerator is greater than the denominator, as in $\frac{7}{5}$, the fractional numerical expression denotes more than one of the things counted. Thus $\frac{7}{5}$ is equal to five fifths, together with two fifths, or to $1\frac{2}{5}$. Different varieties of fractional numerical expressions are distinguished by names as follows:—

A PROPER FRACTION is one whose numerator is less than its denominator.

An IMPROPER FRACTION is one whose numerator is equal to or greater than its denominator. Thus $\frac{3}{8}$ and $\frac{1}{7}$ are called proper fractions, and $\frac{8}{8}$, $\frac{4}{3}$, and $\frac{10}{7}$, are called improper fractions.

A numerical expression consisting of a proper number (often

called a whole number) with a fraction annexed, as $5\frac{3}{7}$, is termed a MIXED NUMBER.*

A SIMPLE FRACTION is one in which the numerator and denominator are simply *proper numbers*, called also *whole numbers.*

A COMPOUND FRACTION is a numerical expression stated in the form of a fraction of a fraction, or in the form of a fraction of a mixed number. Thus $\frac{5}{8}$ is a simple fraction; but $\frac{7}{8}$ of $\frac{10}{11}$, and $\frac{1}{3}$ of $\frac{4}{5}$ of $1\frac{4}{7}$ are compound fractions. Compound fractional expressions have the word *of* interposed between the simple expressions of which they are composed.

A COMPLEX FRACTION is one which has a fraction either in its numerator or denominator or in each of them.

Thus $\frac{5\frac{1}{2}}{9}$, $\frac{8}{9\frac{2}{3}}$, and $\frac{5\frac{1}{2}}{6\frac{2}{3}}$ are called complex fractions. The name *complex fraction* is introduced here in order that it may be noticed in connexion with other names here given; but it is to be observed that the explanations of fractions which have hitherto been given in this treatise do not suffice to indicate what the meaning is of a fractional expression having a fraction in its denominator, such, for instance, as $\frac{8}{9\frac{2}{3}}$. The fractional number here, $9\frac{2}{3}$, occupying the place of the denominator in an ordinary simple fraction, fails in any ordinary language and mode of thought to afford a name or denomination, or to be a "namer," for the parts of a unit, of which parts the numerator 8 states the number taken. Obviously we cannot divide any unit, or anything whatever, into $9\frac{2}{3}$ *equal* parts, and then take 8 of those equal parts. Yet the expression $\frac{8}{9\frac{2}{3}}$ has a perfectly definite and intelligible meaning as a fraction; and the meaning of such expressions will be fully brought out in the more advanced chapter on fractions farther on in this treatise.

The distinctive names, *compound* and *complex*, as applied to certain varieties of fractional numerical expressions more complicated in form than those called *simple fractions*, are mentioned here because of their having been usually given in treatises on arithmetic. The maintaining of these two names for specially distinct meanings seems, however, not at all to simplify or facilitate the subject of fractions to the teacher or the learner; and their separate employment in this way is very properly tending to fall into disuse. Any fractional numerical expressions which are not *simple* might very well be called *complex*.

For brevity, a fractional numerical expression of any form may be called a FRACTIONAL NUMERIC; and we may use the name a NUMERIC to signify in general any numerical expression which is either a *number* properly so called (such as 1, 2, 3, 4 . . . . 9 . . . . 25, &c.), or a fractional numerical expression (such as $\frac{1}{4}$, $\frac{3}{2}$, $5\frac{2}{7}$, $\frac{4}{5}$ of $\frac{2}{3}$, &c.)

Some of the most important properties of fractions will next be stated with explanations.

PROPOSITION 1. If the terms of any fraction be both multiplied

* When the name "numeric" is used as proposed in the foot note on page 4, this may more properly be called a MIXED NUMERIC.

by the same number, the resulting fraction is equivalent to the original one.

To illustrate this, let us multiply the terms of the fraction $\frac{2}{5}$ by 3. We thus get the fraction $\frac{6}{15}$; and this can be seen to be equal to $\frac{2}{5}$ in the following way. If *any thing*—that is to say, *any one whole*—is divided into five equal parts, each is called *one fifth* of it; and if again the same whole is divided into three times as many equal parts—that is, into 15 equal parts—each will have a third of the magnitude that each part had in the former case. That is, $\frac{1}{15}$th * is a third of $\frac{1}{5}$th. Hence $\frac{2}{15}$ must be a third of $\frac{2}{5}$. Hence 3 times $\frac{2}{15}$ must be equal to $\frac{2}{5}$; or, in other words, $\frac{6}{15}$ must be equal to $\frac{2}{5}$. A like illustration may be given in every similar case, and so we may accept the proposition stated above as being established.

Proposition 2. If the terms of any fraction be divided by any number which will divide each without remainder, the resulting fraction is equivalent to the original one.†

The truth of this follows immediately from the preceding proposition. It was there shown that $\frac{2}{5}$ may be changed in form to $\frac{6}{15}$ without alteration of value, or, in other words, that $\frac{2}{5}$ and $\frac{6}{15}$ are equal each to the other. So if we commence now with $\frac{6}{15}$ as the given fraction, and divide its numerator and denominator both by the same number, 3 (which will divide each without a remainder), we get the fraction $\frac{2}{5}$, which, as previously shown, is equivalent to the original one. A similar illustration might be given in every similar case, and so the proposition may be admitted as being true.

Proposition 3. A fraction may be multiplied by any number by multiplying its numerator by that number, or by dividing its denominator by that number, in case of the denominator being divisible by it without remainder.‡

To illustrate this, let us begin with the fraction $\frac{2}{15}$, and let it be required to multiply that fraction by 3. Now, obviously 3 times $\frac{2}{15}$ must be $\frac{6}{15}$, just as 3 times 2 shillings are 6 shillings. But further, this result, $\frac{6}{15}$, will not be altered in value if we divide both its terms by 3, and so we see that it is equal to $\frac{2}{5}$. Thus we have shown that 3 times $\frac{2}{15}$ may be expressed either as $\frac{6}{15}$ or as $\frac{2}{5}$. But the one is got from the original fraction by multiplying the numerator by 3, and the other is got from it by dividing the denominator by 3. Hence we may accept the proposition as being established.

Proposition 4. A fraction may be divided by any number,

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* It is of no importance whether this be written $\frac{1}{15}$th or $\frac{1}{15}$. The letters *th* may sometimes advantageously be annexed, in order to keep the reader in mind to read $\frac{1}{15}$, $\frac{1}{5}$, &c., as one fifteenth, one fifth, &c.

† If the terms of the original fraction were divided by a number which would not divide them both without remainder, the resulting fraction would be a complex one; and although the proposition admits of being extended to such cases, yet in the present introductory chapter on fractions the intention is to treat only of some of the most important fundamental principles, and most useful and easy practical operations, and to leave more intricate considerations for the advanced chapter on fractions farther on in this treatise.

‡ In case of there being a remainder in dividing the denominator by that number, the resulting fraction would be a complex one; and the proposition stated in the text is purposely framed not to include operations with complex fractions, which the pupil may not be prepared to enter on when first working in the present chapter.

either by dividing its numerator by that number, when the numerator is divisible by it without remainder,* or by multiplying the denominator by that number.

For illustration of the first part of this we may observe that if we are to divide $\frac{6}{15}$ by 3, we get $\frac{2}{15}$; just as in dividing £6 by 3 we get £2; and here we see that the fraction $\frac{6}{15}$ is divided by 3 by dividing its numerator by 3. Next, to illustrate the second part of the proposition, we may observe that if we are to divide $\frac{2}{5}$ by 3, we must get the same as when we divide its equal $\frac{6}{15}$ by 3, and that was seen to be $\frac{2}{15}$. That is to say, if we divide $\frac{2}{5}$ by 3, we get $\frac{2}{15}$; so this fraction $\frac{2}{5}$ is divided by 3 by multiplying its denominator by 3. Hence the proposition may be accepted as true.

Some examples and exercises in the application of the principles of fractions which have now been developed will next be given; and a few rules will be introduced at places where they can readily receive illustration from the examples. It may be observed that the four propositions just brought forward, and also some of the previous explanations, might readily be translated into formal rules, or that formal rules for procedure might readily be deduced from them. It is, however, better often to let the learner think for himself how to proceed to work from *principles* clearly put before him, rather than that he should be led to expect for every case a formal *rule*. Working too much by rule tends injuriously towards the neglecting or forgetting of the principles on which the operations depend.

Exam. 1. Find a fraction which shall be equivalent to $\frac{2}{3}$, and shall have its denominator seven times as great as the denominator in that given fraction.

To solve this we have only to notice that, on the principle stated in Proposition 1, if the denominator of the required fraction is to be 7 times the denominator of the given fraction, the numerator of the required fraction must be 7 times the numerator of the given fraction. Hence the required fraction equivalent to the given fraction will be $\frac{14}{21}$.

Exam. 2. Find a fraction which shall be equivalent to $\frac{13}{17}$, and shall have its numerator 23 times as great as the numerator in that given fraction.

Here obviously again, according to Prop. 1, we must multiply the numerator and denominator of the given fraction each by 23, to find the numerator and denominator of the required fraction. Thus the required fraction is found to be $\frac{299}{391}$.

Exam. 3. Multiply the fraction $\frac{5}{12}$ by 2.

Here, according to Prop. 3, we may either multiply the numerator by 2 or divide the denominator by 2. In the one way we get $\frac{10}{12}$, and in the other way we get $\frac{5}{6}$, as the result required. We may notice that these two expressions are equivalent each to the other, because obviously the second could be obtained from the first by dividing the numerator and denominator of the first each by 2, a

* The reason for introducing here the words, "when the numerator is divisible by it without remainder," will be understood from the two preceding notes.

proceeding which, according to Prop. 2, would not alter the value of that fraction.

Exam. 4. Multiply the fraction $\frac{5}{22}$ by 7.

Here, as the denominator 22 is not divisible by 7 without remainder, we may best proceed by multiplying the numerator by 7, and so we find that 7 times $\frac{5}{22}$nds are $\frac{35}{22}$nds.

Exam. 5. Divide the fraction $\frac{35}{43}$rds by 7.

Here, according to Prop. 4, we may either divide the numerator by 7, or we may multiply the denominator by 7. Thus in the one way we get $\frac{5}{43}$, and in the other way we get $\frac{35}{43 \times 7}$, which is $\frac{35}{301}$. So the required fraction may be stated either as $\frac{5}{43}$ or as $\frac{35}{301}$. These two fractions are equivalent each to the other, as may be seen by dividing both numerator and denominator in the second by 7, which will show $\frac{5}{43}$ as the equivalent fraction.

Exam. 6. Divide the fraction $\frac{5}{9}$ by 7.

Here, since the numerator is not divisible by 7 without remainder, we may best proceed by multiplying the denominator by 7; and so we find that one seventh part of $\frac{5}{9}$ths is $\frac{5}{63}$rds.*

Exam. 7. Supposing that the wages of a carpenter working full time, without overtime, for a week are 34 shillings; and that a carpenter, A, has earned in a week 29 shillings: express his wages received, as a fraction of the regular week's wages, 34 shillings, which for brevity may be called the *standard* (or may, if we choose, be regarded as a *unit*, or *one whole quantity;—one regular week's wage*, of which the sum earned by him will be a fraction).

Here we may notice that 1 shilling would be 1 thirty-fourth (or, as it may be denoted, $\frac{1}{34}$th) of the regular weekly wage. Then the carpenter A, in receiving 29 shillings, receives 29 thirty-fourths of that standard, or, as the same may be written, $\frac{29}{34}$ths of the standard.

Exam. 8. Supposing the regular weekly wages of a carpenter to

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* It will be seen, from the words in which the result has been stated, that the requirement to divide $\frac{5}{9}$ths by 7 has been taken to signify a requirement to find one seventh part of $\frac{5}{9}$. This is quite in accordance with one of the objects of DIVISION stated at (2.) on page 32, except that if the wording there given, viz. that one of the objects of division is "*to find a number which will be contained a given number of times in another given number*," were to be applied to the present case, it would be necessary that we should now use the word number in its extended signification as meaning any numeric, whole or fractional, while in that former passage it was used to mean only a number properly so called, or what is often spoken of as a "whole number." The question in the text here in Exam. 6 may properly be understood as meaning—*to find the numeric which will be contained 7 times exactly in $\frac{5}{9}$ths.*

If the question were put in another form, in which questions in division are often put, viz. if it were asked, how many times is 7 contained in $\frac{5}{9}$ths, or what is the *quotient* in dividing $\frac{5}{9}$ths by 7, the question so put would be almost unintelligible; but still it could be interpreted so as to make the answer be, 7 is contained in $\frac{5}{9}$, not twice, nor once, but only $\frac{5}{63}$rds of once, which must be taken to mean that $\frac{5}{63}$rds of 7 is contained exactly once in $\frac{5}{9}$. Fuller explanations on such subjects as this will be given in the more advanced chapter on fractions.

be 34 shillings, as in the last foregoing example, and that a carpenter, B, has earned 158 shillings in a period of four weeks, in which he has worked overtime on some days: express his total wages for the four weeks as a numeric of the standard or regular weekly wage.*

Here 1 shilling would be $\frac{1}{34}$ of the standard, and so the total earnings, 158 shillings, amount to $\frac{158}{34}$ of the standard. This is one statement of the required result. If it be wished to express this improper fraction as a proper number together with a proper fraction, we may find, by dividing 158 by 34, that 34 is contained in 158 four times, with a remainder of 22. Hence 158 thirty-fourths are equal to $\frac{34}{34}$ths taken 4 times, plus $\frac{22}{34}$ths. But $\frac{34}{34}$ths are equal to 1, and 4 times the same are 4. Hence $\frac{158}{34} = 4\frac{22}{34}$. So the carpenter's earnings are $4\frac{22}{34}$ of the standard weekly wage. The fraction $\frac{22}{34}$ in this may, however, be expressed by smaller numbers, since obviously its numerator and denominator may each be divided by 2; and so we find $\frac{11}{17}$ as an equivalent fraction. Hence his total earnings in the four weeks may be stated as $4\frac{11}{17}$ of the standard weekly wage.

The foregoing example, No. 8, includes several operations in fractional arithmetic. The one of these in which it was shown that the improper fraction $\frac{158}{34}$ may be otherwise expressed by an equivalent mixed numeric, $4\frac{22}{34}$, may be specially remarked on here as being an example of what is called the process of *reducing an improper fraction to a whole or mixed numeric.* The explanations given in that operation supply sufficient illustration for the following important rule, stated in wording commonly used.

RULE I. *To reduce an improper fraction to a whole or mixed number.*† Divide the numerator by the denominator; the quotient will be the whole number required, and if there be any remainder, write it over the given denominator for the fractional part of the required result.

Again, the last two foregoing examples, Nos. 7 and 8, supply sufficient illustration for the following important rule.

RULE II. *To express any given number as a fraction of another given number considered as a standard, or as a secondary unit.* Make the standard number the denominator of the required fraction, and the other given number its numerator.‡

REMARK. The process stated in this rule is often used in expressing one given *quantity* as a fraction of another given *quantity*, as when we ask: What fraction is the length 5 yards of the length 30 yards? and answer that the length 5 yards is $\frac{5}{30}$ths, or $\frac{1}{6}$th of the length 30 yards.

Exercises. Reduce the following fractions to lower terms; that is:

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* The requirement in this question would often be stated in either of the two following ways:—(*a.*) Express his total wages as an improper fraction of the regular weekly wage; or (*b.*) express his total wages *in terms of* the regular weekly wage.

† This might preferably now be called *a whole or mixed numeric.*

‡ The word "number" in this rule, as given at the present early stage, may for simplicity be understood as meaning "whole number." In the advanced chapter on fractions, however, it will be shown that the rule may be extended so as to apply alike to any numerics, whether whole or fractional.

to say, find fractions equivalent to them, but expressed by smaller numbers:—*

Exer.	*Answ.*	*Exer.*	*Answ.*	*Exer.*	*Answ.*
1. $\frac{6}{18}$	$\frac{1}{3}$	5. $\frac{5}{100}$	$\frac{1}{20}$	9. $\frac{49}{35}$	$\frac{7}{5}$
2. $\frac{8}{26}$	$\frac{4}{13}$	6. $\frac{16}{12}$	$\frac{4}{3}$	10. $\frac{33}{36}$	$\frac{11}{12}$
3. $\frac{8}{24}$	$\frac{1}{3}$	7. $\frac{13}{130}$	$\frac{1}{10}$	11. $\frac{28}{35}$	$\frac{4}{5}$
4. $\frac{9}{33}$	$\frac{3}{11}$	8. $\frac{84}{96}$	$\frac{7}{8}$	12. $\frac{25}{50}$	$\frac{1}{2}$

Exer.	*Answ.*	*Exer.*	*Answ.*
13. Multiply $\frac{4}{13}$ by 3	$\frac{12}{13}$	16. Divide $\frac{4}{13}$ by 2	$\frac{2}{13}$
14. „ $\frac{5}{24}$ by 4	$\frac{5}{6}$	17. „ $\frac{4}{13}$ by 3	$\frac{4}{39}$
15. „ $\frac{8}{19}$ by 3	$\frac{24}{19}$	18. „ $\frac{5}{23}$ by 6	$\frac{5}{138}$

Reduce the following fractions to "whole or mixed numbers":—

Exer.	*Answ.*	*Exer.*	*Answ.*	*Exer.*	*Answ.*
19. $\frac{19}{19}$	1	21. $\frac{7536}{375}$	$20\frac{36}{375}$ or $20\frac{12}{125}$	23. $\frac{509}{12}$	$42\frac{5}{12}$
20. $\frac{6437}{298}$	$21\frac{179}{298}$	22. $\frac{10000}{111}$	$90\frac{10}{111}$	24. $\frac{750}{25}$	30

25. Express 28 days as a fraction of a year, taken as 365 days; or briefly, express 28 as a fraction of 365. *Answ.* $\frac{28}{365}$ths.

26. Out of a flock of 528 sheep, 32 were drowned in a flood: express the number lost as a fraction of the whole original number of the flock. *Answ.* $\frac{32}{528}$, or $\frac{2}{33}$.

A few examples and exercises will next be given in operations which are called multiplying and dividing by "numbers containing fractions," and each example will be explained here for itself. The giving of general explanations in a more systematic manner in respect to the processes called multiplying and dividing by fractional numbers (or rather fractional numerics) will be reserved for the more advanced chapter on fractions.

Exam. 9. How many seconds are there in $29\frac{1}{2}$ days, the number of seconds in each day being 86,400?

Here, omitting the ciphers, we multiply 864 by 29 in the usual manner; but, before adding, we divide 864 by 2, and, writing the quotient, 432, under the partial products, adding the three lines together, and annexing two ciphers, we find that the required number of seconds is 2,548,800.

$$\begin{array}{r} 86400 \\ \underline{29\frac{1}{2}} \\ 7776 \\ 1728 \\ \underline{432} \\ 2548800 \end{array}$$

The answer would also be found by doubling $29\frac{1}{2}$, which gives 59; and then multiplying 86400 by 59, and taking half the product.

Exam. 10. If the circumference of a carriage wheel be $14\frac{2}{3}$ feet, how often will it turn round in going a mile, the mile containing 5280 feet?

* In the very easy set of exercises in reducing fractions to lower terms offered here for solution, the pupil can scarcely fail to notice intuitively suitable numbers by which to divide the numerators and denominators. In the advanced chapter on fractions it will be shown how to make sure of reducing any fraction to its lowest possible terms by use of what is called the *greatest common measure* of the numerator and denominator.

Here obviously we want to find how many circumferences of the wheel there are in 5280 feet, as that will be the number of turns of the wheel in going a mile. Hence we have to divide 5280 by $14\frac{2}{3}$. Then, since the denominator in the fractional part of the divisor is 3, we treble both the divisor and the dividend, thus obtaining 44 and 15840; and then, by dividing the latter by the former, we get 360, the number of turns required.

$$\begin{array}{r|l} 14\frac{2}{3}) & 5280 \\ 3 & \quad 3 \\ \hline 44\ ) & \overline{15840}\ (360 \end{array}$$

Exam. 11. How many perches are there in 1000 yards, each perch consisting of $5\frac{1}{2}$ yards?

Here we have to find how often the length one perch, or $5\frac{1}{2}$ yards, or 11 half-yards, is contained in the length 1000 yards, or 2000 half-yards. By division we find that 11 is contained in 2000 181 times, with a remainder 9; and so in the case before us we see that the length 2000 half-yards contains 11 half-yards 181 times, and contains 9 half-yards besides; or, in other words, that the length 1000 yards contains exactly 181 perches and $4\frac{1}{2}$ yards.

After a little experience the mode of procedure to be adopted for the solution of any such questions as this may come to be perceived almost without consideration; but it is important sometimes to bring the nature and meaning of such processes very clearly before the mind, as it often happens that persons work out questions practically with but a vague notion of the meaning of the units they are using, and of the remainders or fractions which they write down.

Exercises.	*Answers.*	*Exercises.*	*Answers.*
27. $27346 \times 2\frac{1}{2}$......	$= 68365$	31. $477121 \times 1\frac{1}{2}$......	$= 715681\frac{1}{2}$
28. $41785 \div 2\frac{1}{2}$......	$= 16714$	32. $477121 \div 1\frac{1}{2}$......	$= 318080\frac{1}{2}$
29. $24579 \div 12\frac{4}{5}$......	$= 1920\frac{15}{64}$	33. $4275 \times 4\frac{1}{4}$......	$= 18168\frac{3}{4}$
30. $25974 \div 17\frac{1}{2}$......	$= 1484\frac{8}{35}$	34. $24248 \div 2\frac{3}{7}$......	$= 9984\frac{8}{17}$

TABLES OF MONEY, WEIGHTS, MEASURES, &c.*

TABLE OF MONEY.

4 farthings............................	= 1	penny......*d.*†
12 pence.................................	= 1	shilling...*s.* or /
20 shillings..............................	= 1	pound or sovereign, £.

A pound contains 240 pence, 480 halfpence, or 960 farthings.

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* Before the pupil proceeds to reduction, the compound rules, &c., he should be acquainted with the divisions of money, weights, and measures, which are most generally used. The teacher will exercise his own judgment in determining which of the tables here given should be committed to memory. The tables fixed on for this purpose should be severally committed, immediately before the pupil commences those parts of reduction in which they are respectively employed; and when he commences the corresponding parts of the compound rules, it will have a good effect to require him to revise them.

† This character, and others similarly placed, in this and some of the following tables, are used as abbreviations for the names that precede them. The

STANDARD OF WEIGHT OR MASS.*

By Act of Parliament 18th and 19th of Victoria, Chapter 72, July 30, 1855, it is enacted that a certain piece of platinum, referred to as a "Weight of Platinum," which had previously to the Act been prepared with extraordinary scientific precautions, and is particularly described in the Act, and stated to be deposited in the

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marks of abbreviation are generally omitted when they are sufficiently understood without explanation.

Farthings were formerly denoted by *q.*; but now $\frac{1}{4}$, annexed to pence, denotes a farthing; $\frac{1}{2}$, a halfpenny, or 2 farthings; and $\frac{3}{4}$, 3 farthings. £. *s.* *d.* and *q.* are the initial letters of the Latin words, *libra*, *solidus*, *denarius*, and *quadrans*, which denote *pound*, *shilling*, *penny*, and *farthing*, or *quarter*, respectively. The character / is a corruption of the long *ſ*, arising from rapidity in making it.

Farthing comes from the Saxon *feorthung*, from *feower*, four, and *feorth*, fourth.

Penny is derived from a weight of same name, since called for distinction a penny-weight. Silver pennies, both in weight and name, were amongst the earliest English coins.

Shilling seems to have been the name of a weight. (Smith's *Wealth of Nations*, Book 1, Chap. 4.)

Pound in account is derived from a pound weight. From the Conquest till Edward I. the English pound, or 20 shillings, contained a pound Tower weight of silver, 37 parts pure to 3 parts alloy—the Tower pound being 11 oz. 5 dwt. troy; whence it can be shown, by an easy calculation, that a pound Troy would be coined into $21\frac{1}{3}$ shillings; or rather, as is quantitatively equivalent, that three pounds Troy would be coined into 64 shillings.

The quantity of silver in the coins was diminished from time to time till the 43rd Elizabeth, when a pound troy was coined into 62 shillings instead of $21\frac{1}{3}$. English pounds, shillings, and pence were thus depreciated, from Edward I. to Elizabeth, to about one third their previous weight and value.

In 1774 gold became the standard of value in Great Britain, 40 lbs. troy of gold (11 parts pure to 1 part alloy) being coined into 1869 sovereigns. Ever since the pound of account has meant the value of a sovereign, or of 5 dwt. $3\frac{171}{623}$ grains of gold of the above standard. In 1816 silver coins were reduced in weight and value, a pound troy being coined into 66 shillings instead of 62, so that a shilling now weighs 3 dwt. $15\frac{3}{11}$ grains.

By the discovery of very fertile gold and silver mines in the sixteenth century, both gold and silver fell in value as compared with corn, from 1570 to 1640, to less than one third their former value, so that pounds, shillings, and pence, before the recent Californian and Australian discoveries, were worth in corn less than a third what they were in the reign of Queen Elizabeth, and, from the change in weight above referred to, less than one ninth what they were in the reign of Edward I.

Other coins are the guinea, value 21*s.*; half-guinea, 10*s.* 6*d.*; the crown, 5*s.*; the half-crown, 2*s.* 6*d.*; the florin, 2*s.*; the noble, 6*s.* 8*d.*; the angel, 10*s.*; the mark, or merk, 13*s.* 4*d.*; the pistole, about 16*s.* 10*d.*; the moidore, 27*s.*; and the groat, 4*d.* Most of these are no longer in use.

From January 1, 1826, the currency of Ireland became assimilated to that of Great Britain. In all leases, contracts, and accounts prior to that date the Irish pounds, shillings, pence, and farthings were of less value than those of Britain, 13 of each denomination in the former country being equivalent only to 12 of the corresponding denomination in the latter. Thus, the British shilling was equivalent to 13 pence Irish; the British pound or sovereign to £1 1*s.* 8*d.* Irish; the Irish shilling being equal to about 11*d.* British, and the Irish pound to about 18*s.* $5\frac{1}{2}$*d.* British.

* The word *weight* unfortunately has two meanings in frequent use, and this ambiguity is a troublesome source of perplexity, especially among students of Natural Philosophy. Firstly, the word very commonly signifies *quantity of matter*, as when we speak of a hundred-weight of iron, or of four pounds weight of sugar; or when, in an Act of Parliament, a certain piece of platinum, or of brass, is carefully and authoritatively designated as a "Pound Weight;" and

Office of the Exchequer, shall be denominated the IMPERIAL STANDARD POUND AVOIRDUPOIS, and shall be deemed to be the only standard measure of weight from which all other weights and other measures having reference to weight shall be derived, computed, and ascertained; and one seven thousandth part of such pound avoirdupois shall be a GRAIN, and five thousand seven hundred and sixty such grains shall be deemed to be a POUND TROY. The Act also provides for the careful preservation, in various places, of copies of this Standard Pound, made with extreme accuracy; and for the restoration of the same standard quantity of weight or of mass by an equivalent piece, in case of the loss or destruction, in any way whatsoever, of the original standard piece.

TABLE OF AVOIRDUPOIS WEIGHT.

7000 grains = 1 pound.

16 drams...... = 1 ounce, oz.	14 pounds......... = 1 stone.
16 ounces...... = 1 pound, lb.	8 stone............ = 1 cwt.*

28 pounds............... = 1 quarter of a hundred-wt., q. or qr.
4 quarters, or 112 lbs. = 1 hundred-weight.........cwt. or c.
20 hundred-weights... = 1 ton.

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secondly, the same word *weight* is often used to signify the downward force which a piece of matter exerts on whatever supports or suspends it, as when we speak of the *weight* of a piece of matter hung to a spring balance as being the *force* which draws out the spring.

The word MASS is employed to express distinctly, in scientific language, one of the two meanings of the word "weight"—that, namely, in which it signifies *quantity of matter*; but for the other meaning in which it signifies a *force*, we have no established name as yet: it may, however, very well be called the *gravity* of the piece of matter.

Now it ought to be known that the downward force of a piece of matter, or its *gravity*, which constitutes one of the two meanings of the word "weight," is different at different latitudes on the earth's surface, being least at the equator, and increasing as we pass from the equator towards either pole, and diminishing as we ascend any high mountain. These variations in the gravity of any piece of matter carried to different places accessible to men, although they are so small in amount as not to be perceptible in ordinary weighings with spring balances, are very important in principle. While the *gravity* or downward force of any piece of matter is variable for different places, its *mass*, or the quantity of matter contained in the piece, is essentially the same everywhere.

* The hundred-weight here mentioned is sometimes called the *great* hundred, or the *standard* hundred, to distinguish it from hundreds of different magnitudes, which are used in particular places. One of the most general of these is the *long* hundred, or the hundred in *long* weight, which contains 120 pounds: hence,

In Long Weight, 30 pounds avoirdupois.... = 1 quarter;
4 quarters, or 120 pounds = 1 hundred-weight.

The use of this weight, however, is now illegal.

The stone, in the greater number of places, is 14 pounds, which alone is the legal one; but in different parts of England it is of various magnitudes, from 8 to 16 pounds. In Ireland also, in the sale of some articles, the stone of 16 pounds, or one seventh of the standard hundred, is used. A ton of stones is 21 hundreds, *long* weight.

In England, 14 pounds of wool = 1 stone, 2 stone = 1 tod, 6 tods and a half = 1 wey, 2 weys = 1 sack, 12 sacks = 1 last; and a pack of wool = 240 pounds.

The sooner that these various local weights be totally abandoned from practical use the better: but it is still desirable that knowledge of them should be preserved because of their use in the past, if for no other reason.

TABLE OF TROY WEIGHT.*

24 grains	= 1 penny-weight,	dwt.
20 penny-weights	= 1 ounce,	oz.
12 ounces	= 1 pound	lb.

It is to be borne in mind that, while the grain is one definitely fixed weight, being specified as $\frac{1}{7000}$th part of the Imperial Standard Pound Avoirdupois, the troy ounce and pound are different weights from the avoirdupois ounce and pound. The troy ounce contains 480 grains, and the avoirdupois ounce contains $522\frac{1}{2}$ grains, and the troy pound is $\frac{5760}{7000}$ of the avoirdupois pound.

STANDARD OF LENGTH.

By the Act of Parliament already referred to, 18th and 19th of Victoria, Chapter 72, July 30, 1855, it is enacted that the distance between the centres of the two gold plugs in a certain bronze bar, which had been prepared with extreme scientific care before the passing of the Act, and is accurately described in the Act, and stated to be deposited in the Office of the Exchequer, shall, when the temperature of the bronze bar is sixty-two degrees by Fahrenheit's thermometer, be deemed to be the IMPERIAL STANDARD YARD.

TABLE OF LONG OR LINEAL MEASURE.

12 lines = 1 inch†	40 perches = 1 furlong
12 inches = 1 foot	8 furlongs = 1 mile §
3 feet = 1 yard	3 miles = 1 league.
$5\frac{1}{2}$ yards = 1 perch‡	

* Troy weight was introduced into Europe from Cairo, in Egypt, about the time of the crusades, and was first adopted in Troyes, a city in France, where great fairs were held, and whence it has its name.

This weight was formerly used for weighing articles of every kind; it is now employed in weighing gold, silver, jewels, and liquors; and grains counted in any numbers large or small, without the use of penny-weights or ounces are employed in philosophical experiments, but the French grams and milligrams are rapidly coming into use in their stead.

Troy weight is also employed by apothecaries in mixing their medicines, though they buy and sell them by avoirdupois weight. When troy weight is thus used it is called APOTHECARIES' WEIGHT; but in this case the ounce (℥) is divided into 8 *drams* (ʒ), the dram into 3 *scruples* (℈), and the scruple into 20 *grains*.

† 3 *barleycorns* make an inch. The barleycorn, however, is never employed now as a measure. Instead, also, of being divided into lines, the inch is now generally divided into eighths, sixteenths, and thirty-seconds, or into tenths and hundredths.

‡ The perch is also sometimes called a *pole*, or *rod*. Each of the names given to this measure is expressive of the instrument by which it was formerly measured—a *rod*, a *pole*, or *perche*, a French word of the same import. In some counties of England the perch is 6 yards, in some 7 yards, and in others 8 yards. In *Cunningham measure* it is $6\frac{1}{4}$ yards; in *Forest measure*, 8 yards; and in *Woodland*, or *Burleigh measure*, 6 yards. None of these, however, is now legal. The yard is said to have been taken from the length of the arm of Henry I. of England.

§ From this table, by an easy reduction, it will appear that a mile contains 320 perches, 1760 yards, or 5280 feet.

Also 4 perches or 66 feet = 1 chain = 100 links; and 80 chains = 1 mile.

A *fathom* is 2 yards, or 6 feet; a *pace*, 5 feet; a *hand* (used in measuring horses), 4 inches; a *span*, 9 inches, and a nautical or geographical mile, by Admiralty Regulation, is the length of one minute of longitude at the equator, and is ascertained to be about 6086 feet, or approximately 1000 fathoms. In nautical language $\frac{1}{10}$th of a nautical mile is sometimes called a cable, and in connexion with this, $\frac{1}{100}$th of a cable is called a fathom, being nearly equal to a true fathom.

TABLE OF CLOTH MEASURE.

4 nails......... = 1 quarter | 4 quarters...... = 1 yard.*

A Flemish ell is 3 quarters of a yard; an English ell, 5 quarters, or a yard and a quarter; and a French ell, 6 quarters, or a yard and a half.

TABLE OF SQUARE MEASURE, OR MEASURE OF SURFACE.

144 square† inches	= 1 square foot
9 square feet....................................	= 1 square yard
$30\frac{1}{4}$ square yards................................	= 1 square perch.

TABLE OF AREAS IN LAND MEASURE.

40 square perches.................................	= 1 rood
4 roods, or 160 square perches................	= 1 acre
10 square chains, or 100,000 square links	= 1 acre.‡

Till the year 1826, the perch in Ireland contained 7 yards instead of $5\frac{1}{2}$, so that 11 Irish miles were equivalent to 14 British ones, and the Irish mile contained 2240 yards, or 6720 feet.

* Cloth measure is a species of long measure, and the yard is the same in both. Hence, a quarter of a yard is 9 inches, and a nail $2\frac{1}{4}$ inches.

† A *square* is a figure which has four equal sides, each perpendicular to the adjacent ones. A *square inch* is a square, each of whose sides is an inch in length; a *square yard*, a square, each of whose sides is a yard in length, &c. The table of square measure is formed from the table of long measure by multiplying the number there belonging to each lineal dimension by itself, as in the following example:—A square foot is $= 12 \times 12 = 144$ square inches, &c.

‡ In measuring land, surveyors use the *chain*, which, as is stated above, is 4 perches in length, and is divided into 100 equal parts, called *links*. They also compute by square chains and square links, but exhibit the result in acres, roods, and perches. Four lineal perches in Imperial or English statute measure being equivalent to 792 inches, it follows, from dividing by 100, that the length of a link is $7\frac{92}{100}$ inches. It may be observed, also, that 640 acres are a *square mile*; and that a *hide* of land, mentioned by old writers, is 100 acres.

The relations among the chief land measures which have been, or are still, in use in England, Scotland, and Ireland, may be stated as follows:—

One Imperial acre = $\frac{121}{196}$ of the acre in Irish Plantation Measure.

One Imperial acre = $\frac{1089}{1369}$ of the Scotch acre.

One Imperial acre = $\frac{484}{625}$ of the acre in Cunningham Measure.

One Imperial acre = $\frac{121}{144}$ of the acre in Woodland or Burleigh Measure.

One Imperial acre = $\frac{121}{256}$ of the acre in Forest Measure.

TABLE OF CUBIC OR SOLID MEASURE, OR MEASURE OF VOLUME.

1728 cubic * inches	= 1 cubic foot
27 cubic feet	= 1 cubic yard.

TABLE OF LIQUID AND DRY MEASURE.†

4 gills	= 1 pint	2 gallons	= 1 peck
2 pints	= 1 quart	4 pecks, or 8 gallons	= 1 bushel
4 quarts, or 8 pints	= 1 gallon	8 bushels	= 1 quarter.

TABLE OF TIME.

60 seconds	= 1 minute
60 minutes	= 1 hour
24 hours	= 1 day
7 days	= 1 week
52 weeks and 1 day, or 365 days	= 1 common year
52 weeks and 2 days, or 366 days	= 1 leap year.‡

For the use of readers who may be already acquainted with proportion, and reduction of fractions, and also with some of the principles of mensuration, it may here be explained that the magnitudes of the acres in the different Land Measures are proportional to the squares of the lineal perches in those Land Measures. Thus, on reference to the Table of Long Measure on page 58, and to one of the foot notes annexed to that Table, it will be seen that an Imperial perch lineal is 11 half-yards, and a lineal perch in Irish Plantation Measure is 14 half-yards; and then, by taking the squares of those two numbers, we have :—

one Imperial acre : one Irish acre : : 121 : 196,

and hence,

one Imperial acre = $\frac{121}{196}$ of the Irish acre;

and the numerical relations in the other cases may be brought out in like manner, with the aid of processes in reduction of fractions when necessary. In order to prevent a common mistake, it may be proper to remark that the difference in the comparative magnitudes of the *acres* is much greater than that of the *lineal perches*. Thus, while 121 Irish are equivalent to 196 English *square* perches, 121 Irish *lineal* perches are equivalent to only 154 English ones. The chain in Scotland, before 1826, was fixed at 74 feet; and hence, since the Imperial chain is 66 feet, we may find by comparison of square chains, which will give the same result as a comparison of square perches, that

one Imperial acre : one Scotch acre : : 66×66 : 74×74;

and hence that

one Imperial acre is $\frac{1089}{1369}$ of a Scotch acre.

It may be observed that now the only *legal* measure for land is the English statute measure, called also Imperial measure.

* A *cube* is a figure contained by six equal squares. Dice afford a familiar instance of this figure. A *cubic inch* is a cube whose faces are each a square inch; a *cubic foot*, a cube whose faces are each a square foot, &c. It may be remarked that 1728 is equal to $12 \times 12 \times 12$, and 27 to $3 \times 3 \times 3$.

† By this measure, which is evidently a species of solid measure or measure of volume, liquids, and also grain and other dry goods are sold. The peck, bushel, and quarter are used only for dry goods. In reference to such goods, also, the *wey* or *load*, containing 5 quarters, and the *last*, 10 quarters, are sometimes spoken of: but the sooner that the number of various units used in what is called "Liquid and Dry Measure," or generally in Measure of Volume, is greatly diminished, the better.

The use of *heaped measure* was done away by Act of Parliament in 1835.

‡ The year is divided into 12 portions, called *calendar months*, the names of which are January, February, March, April, May, June, July, August, Sep-

TABLE OF ANGULAR MEASURE, OR OF ANGULAR DIVISION OF THE CIRCLE.

An *angle* is the opening between two straight lines which meet. The point in which they meet is called the *vertex* of the angle; and the lines radiating out from that point may be spoken of as *radial lines.*

If a straight line is kept constantly in a plane, and with one end centred at a fixed point, and is made to revolve with a motion like that of the hand of a clock, it moves through an angular space called a *round*, in turning one revolution.

One fourth of a round is called a *right-angle*, or a *quarter-round.*
One three hundred and sixtieth of a round is called a *degree.*
One sixtieth of a degree is called a *minute.*
One sixtieth of a minute is called a *second.*

Right-angles, degrees, minutes, and seconds are often considered as marked out by spaces on the circumference of a circle having any radius, and described round the vertex of the angle as centre.* Thus, if the circumference is divided into 360 equal parts, each of these shows a *degree* of angular space round the centre, and is often spoken of as a degree on that circumference. If a degree space on the circumference is divided into 60 equal parts, each of these parts shows a minute of angular space round the centre. If a minute space on the circumference is divided into 60 equal parts, each of these parts shows a *second* of angular space round the centre.†

---

tember, October, November, December. Of these, April, June, September, and November have 30 days each; and the rest, except February, have 31 days each. In leap years, February has 29 days; in common years, 28 days; so that a leap year contains 366 days, and any other 365. The precise length of the year is found to be 365 days, 5 hours, 48 minutes, 48 seconds; it is, therefore, 365 days, 6 hours, nearly.

Leap years occur at intervals of 4 years, and may be known by dividing by 4 the number expressed by the last two figures in the number of the year, according to the Christian era: if there be no remainder, it is a leap year; otherwise, the remainder shows how many years it is after leap year. To this there is one exception, as the exact centuries are not leap years, except when the number of centuries is divisible by 4, without remainder. Thus, the year 1840 was a leap year, because 40 is divisible by 4; but 1839 was the third year after leap year, because 3 remain when 39 is divided by 4. Also, the year 2000 will be a leap year, but 1900 not, as 20 is divisible by 4, but 19 not. Hence, in 400 years there are 97 leap years.

Learners may easily remember the number of days contained in each month by recollecting that *the months are long and short alternately*, with the exception of August, which, as well as July, is *long*, while the months after it follow the rule.

* A *circle* is a plane figure bounded by one curved line, which line is such that all straight lines drawn to it from a certain point within the figure are equal. That point within the figure is called the *centre* of the circle. The bounding line is properly called the *circumference* of the circle. The name "circle," however, has come often to be applied to the circumference, so that it now sometimes means the bounding line, and sometimes the plane figure or space enclosed by that line. A *radius* of a circle is any straight line drawn from the centre to the circumference. The plural of the word radius is *radii*. A *diameter* of a circle is any straight line passing through the centre and terminated both ways by the circumference. An *arc* of a circle is any part of the circumference.

† Degrees, minutes, and seconds are marked thus: °, ′, ″. Hence, the expression, 41° 24′ 54″, is read, 41 degrees, 24 minutes, 54 seconds. The reason of these marks being employed will appear evident from the consideration that minutes and seconds are only abbreviated expressions for *first* minutes, or minutes of the first order, and *second* minutes, or minutes of the second order; *minutes*, in each instance, signifying *small parts*. It may also be remarked that seconds, both in time and in the circle, were formerly divided each into 60 *thirds*, but that they are now divided into tenths and hundredths.

The angle between two radii of a circle, which comprise between them an arc of the circle equal in length to the radius, is a very important unit of angular measurement, and is called a *radian*. The radian is not commensurable with the round, quarter-round, degree, minute, or second. Its relation to these other units may, however, be given almost quite exactly by stating that:—

One radian is = 206265 seconds, approximately, this being true to the nearest second; or that:—

One radian is = 3438 minutes, approximately, this being true to the nearest minute:—or otherwise, by stating, in a decimal fractional expression, that:—

$$\text{One radian is} = \left\{ \begin{array}{l} 57{\cdot}2958 \text{ degrees} \\ \text{or } 57\tfrac{2958}{10000} \text{ degrees} \end{array} \right\} \text{approximately; or that}$$

$$\text{One half-round is} = \left\{ \begin{array}{l} 3{\cdot}1416 \text{ radians} \\ \text{or } 3\tfrac{1416}{10000} \text{ radians} \end{array} \right\} \text{approximately.}$$

MISCELLANEOUS TABLE.

12 articles......... = 1 dozen	6 score or 120..... = 1 great, or long hundred
12 dozen........... = 1 gross	
20 articles......... = 1 score	24 sheets of paper = 1 quire
5 score............ = 1 hundred	20 quires............ = 1 ream.

TABLES OF FRENCH WEIGHTS AND MEASURES ON THE METRIC SYSTEM will be found farther on in this treatise in the chapter entitled METRIC SYSTEM OF WEIGHTS AND MEASURES. It is a very important system, not in France alone, but throughout the world. By an Act of Parliament entitled the "Metric Weights and Measures Act, 1864," its use is legalized in Great Britain and Ireland. The subject can be better treated after the teaching of Decimal Fractions than at the present stage, and it is therefore deferred.

REDUCTION.

REDUCTION is the process of changing the numerical expression of a quantity, from its numerical expression by units of any one denomination, or set of denominations, to its numerical expression by units of any other denomination, or set of denominations.

RULE I. *To reduce a quantity in one denomination to a lower denomination.* Multiply the number which expresses the quantity in the higher denomination by the number of units of the lower denomination which make one unit of the higher denomination.

RULE II. *To reduce a quantity expressed in units of two denominations to its lower denomination.* Multiply the given number of units of the higher denomination by the number of units of the lower denomination which make one unit of the higher, and add to the product the given number of units of the lower denomination.

Remark supplementary to Rules I. and II.—By successive applications of Rule II., or of that rule and Rule I., a quantity expressed in any number of denominations may have its several parts, which are in different denominations, reduced to any lower denomination, except any part already existing in the lowest denomination available. Also when there are intermediate denominations between a higher denomination in which a quantity is originally expressed and a lower one to which it is required to be reduced, the work, instead of being performed directly in a single operation by Rule I., may be performed by successive applications of Rule I., each operation finding an expression for the quantity as reduced to the next denomination downwards, till the required denomination is reached. Such processes need not be specially detailed in rules, as they will become sufficiently obvious after study of the examples given below.

RULE III. *To reduce a quantity in one denomination to a higher denomination.* Divide the number which expresses the quantity in the lower denomination by the number of units of that denomination which make one unit of the higher denomination; the quotient will be the whole or a large part of the given quantity reduced to the higher denomination, and if there be a remainder, the usual understanding is that it is to be left in its original denomination, and is so to be annexed as an addition to the part of the given quantity which has been reduced to the higher denomination; but, when desired, the remainder may be reduced to a fraction in the higher denomination by writing it as a numerator in a fraction having as denominator the number already taken as divisor.

Remark supplementary to Rule III.—When there are intermediate denominations between a lower denomination in which a quantity is given and a higher one to which it is to be reduced, Rule III. admits of being successively applied so as to give in none of the several denominations a remainder as great as a single unit in the next higher denomination. Such operations, and other varieties of processes in reduction, will be readily understood after the working out and studying of a few examples without further preliminary explanations here.

Reduction from higher to lower denomination as in Rules I.

and II., and their supplementary remark, is often called **reduction descending**; while reduction from lower to higher denominations, as in Rule III. and its supplementary remark, is called **reduction ascending.**

Methods of Proof.

1. Reduce the quantity found in the result back to the denomination or denominations in which the originally given quantity was expressed; and, if the original quantity is thus obtained again, the work is correct.

2. Various steps of the process may be proved separately by the methods of proving multiplication or division.

REDUCTION OF MONEY.

Exam. 1. Reduce £59 to farthings.

In this example the number of pounds is multiplied by 20, to reduce the pounds to shillings, because there are 20 shillings in each pound. The number of shillings, in like manner, is multiplied by 12, to reduce the shillings to pence, and the number of pence by 4, to reduce the pence to farthings, because there are 12 pence in each shilling, and 4 farthings in each penny. Hence it appears that in £59 there are 1180 shillings, 14160 pence, or 56640 farthings. The operation is proved by dividing successively by 4, 12, and 20, the former multipliers, in a reversed order; and the work is correct, since the final quotient, 59, which is the number of pounds found on reducing backwards, agrees with the quantity given in the question.*

£	
59	
20	
1180	shillings
12	
14160	pence
4	
4)56640	far. *answ.*
12)14160	pence
20)1180	shillings
£59,	*proof.*

Exam. 2. Reduce £94 - 12 - 8¼ to farthings.

In this example, in the multiplication by 20, 12 shillings are added to the product; in the multiplication by 12, 8 pence are added; and in the multiplication by 4, 1 farthing is taken in. Hence the answer is 90849 farthings.

£	s.	d.
94	12	8¼
20		
1892	shillings	
12		
22712	pence	
4		
90849	farthings, *answ.*	

Farthings.	
4)90849	
12)22712¼	
20)1892	8¼
£94 - 12 - 8¼,	
proof.	

* The pupil is attentively to observe that the multiplications and divisions in reduction are multiplications or divisions not of quantities, but of numbers. Thus, in the example above, in the multiplication by 12, we do not really multiply 1180 shillings by 12, as in doing that we should get 14160 shillings; but we really mean that 12 times the *number* of shillings is the *number* of pence constituting the same quantity of money; and, for instance, again, in the proof, in the division by 20, we do not take a twentieth of 1180 shillings, which would be 59 shillings, but we do take a twentieth of the *number* of shillings to find the *number* of pounds having the same value.

Exam. 3. Reduce 83918 farthings to pounds, &c.

In this example the number of farthings is divided by 4, because in the same quantity of money there are 4 times as many farthings as there are pence. For a similar reason the number of pence is divided by 12, to reduce the pence to shillings, and the number of shillings thus found, by 20, to reduce the shillings to pounds. Hence, it appears that 83918 farthings are equivalent to 20979 pence, with two farthings, or a halfpenny; to 1748 shillings, and 3 pence halfpenny; or, finally, to £87 - 8 - $3\frac{1}{2}$. The operation would be proved by reducing the answer to farthings, in the manner exhibited in the foregoing example. By this means we should obtain 83918; and this being the same as the given number of farthings, it would follow that the answer is correct.

$$
\begin{array}{r|l}
4 & 8\,3\,9\,1\,8 \text{ farthings} \\ \hline
12 & 2\,0\,9\,7\,9\tfrac{1}{2} \\ \hline
2,0 & 1\,7\,4,8 \quad 3\tfrac{1}{2} \\ \hline
 & \pounds 8\,7 - 8 - 3\tfrac{1}{2}, \textit{answ.}
\end{array}
$$

	Exercises.	*Answers.*
1.	Reduce £341 - 0 - 4 to pence	81844
2.	——— £97 - 17 - $3\frac{1}{2}$ to halfpence	46975
3.	——— £783 - 2 - $3\frac{1}{4}$ to farthings	751789
4.	——— £481 to pence	115440
5.	——— 33333 pence to pounds, &c.......	£138 - 17 - 9
6.	——— £1023 - 16 - 0 to shillings......	20476
7.	——— £113 - 15 - 3 to pence...........	27303
8.	——— £1 - 2 - 9 to pence	273
9.	——— 2300 pence to pounds, &c.........	£9 - 11 - 8
10.	——— £1 - 9 - 3 to pence	351
11.	——— £463 - 19 - $7\frac{1}{2}$ to farthings......	445422
12.	——— £1 - 14 - $1\frac{1}{2}$ to farthings.........	1638
13.	——— 95283 halfpence to pounds, &c.	£198 - 10 - $1\frac{1}{2}$
14.	——— £38 - 14 - 0 to pence	9288
15.	——— £133 - 6 - 8 to farthings.........	128000
16.	——— 47589 pence to pounds, &c.......	£198 - 5 - 9
17.	——— 1234567 farthings to pounds, &c.	£1286 - 0 - $1\frac{3}{4}$
18.	——— £53 - 14 - 0 to farthings.........	51552
19.	——— 75396 shillings to pounds, &c....	£3769 - 16 - 0
20.	——— 13*s.* 4*d.* to pence	160
21.	——— 967 guineas to pounds, &c.*......	£1015 - 7 - 0
22.	——— £497 - 12 - 8 to farthings	477728
23.	——— 69173 pence to guineas, &c.......	274 gs. and 10*s* 5*d.*
24.	——— £1000 to guineas, &c.	952 gs. and 8*s.*
25.	——— £371 - 14 - 10 to halfpence......	178436
26.	——— £42 - 12 - $7\frac{1}{2}$ to halfpence	20463
27.	——— 31 guineas and a half to pence...	7938
28.	——— 3650 pence to pounds, &c.........	£15 - 4 - 2
29.	——— 3120 pence to pounds, &c.........	£13 - 0 - 0

* To reduce guineas to pounds, take as many pounds as there are guineas, and as many shillings besides, and add these together, carrying from the shillings to the pounds one pound instead of every twenty shillings. To reduce pounds to guineas, reduce them to shillings, and divide the number of shillings by 21.

REDUCTION OF AVOIRDUPOIS WEIGHT.

Exam. 6. Reduce 27 cwt. 2 qrs. 22 lbs. to pounds.

In this example we multiply the number of hundred-weights by 4, to find an equivalent number of quarters, because there are 4 quarters in each hundred-weight; and we add in the 2 quarters given as part of the whole original weight. Then the number of quarters, 110, so found, we multiply by 28, to find an equivalent number of pounds, and we add in the 22 pounds given as a part of the whole original weight; and so we find 3102 as the number of pounds which are equivalent to the whole original weight.

cwt. qrs. lbs.
27 2 22
4
110
28
902
220
3102 pounds, *answ.*

lbs.
28 { 4)3102
7)775 2
4)110 22
cwt. 27 - 2 - 22
proof.

RULE IV. *Weights stated in hundred-weights, quarters, and pounds may be very easily reduced to pounds by the following method:*—Multiply the number of hundred-weights by 112, adding in along with the product the number of pounds in the stated quarters, and also the stated number of pounds: recollecting that for 1 quarter, 28 pounds are to be added; for 2 quarters, 56 pounds are to be added; and for 3 quarters, 84 pounds are to be added.

Exam. 7. Find by this method the answer to the question of the last foregoing example; that is, reduce 27 cwt. 2 qrs. 22 lbs. to pounds.

Here 112 is set down as a multiplier. under the number of hundred-weights, 27. Then 56 is set down to be added in as the number of pounds in the stated 2 qrs. Then the stated number of pounds, 22, is set down for addition. Then the three partial products by the figures of 112 are set down so as to range properly for addition; and the sum found, 3102, is the number of pounds required. The reasons are obvious, since there are 112 lbs. in each hundred-weight.

cwt. qrs. lbs.
27 2 22
112
56
22
54
27
27
3102 lbs., *answ.*

Introductory note to Rule V.—This rule serves the same use as Rule IV., but affords a briefer process, which may be preferred by persons who have frequently to make such calculations as are dealt with in these rules.

RULE V. *To reduce hundred-weights, quarters, and pounds, to pounds:*—Multiply the number of hundred-weights by 12, without writing the multiplier, and set the product below the hundred-weights two places to the right hand; and, under the last figures

of that product, place for addition the number of pounds in the stated quarters, and also the stated number of pounds. Then add together all the numbers so arranged, the uppermost one being taken as if it had been formed by annexing two ciphers to the number of hundred-weights.

The *reason for this rule* will be obvious when it is noticed that the uppermost two lines, taken together, would make up the product which would be obtained by multiplying the number of hundred-weights by 112.

Exam. 8. Work out by the method in Rule V. the answer to the same question as was set in Examples 6 and 7.

Here the number of hundred-weights, 27, is multiplied by 12, and the product, 324, is written under the 27 two places to the right for addition, as if two ciphers were annexed to the 27 to multiply it by 100. Then 56, the number of pounds in the stated 2 quarters, is placed below the 324, as is also the stated number of pounds, 22. Then the numbers so placed are added together; and their sum, 3102, is the required number of pounds, which are equivalent to 27 cwt. 2. qrs. 22 ℔s.

cwt.	qrs.	℔s.
27	2	22
324		
56		
22		
3102 ℔s., *answ.*		

Exam. 9. Reduce 591241 pounds to tons, &c.

Here the number of pounds is divided by 28, by means of the factors 4 and 7. The quotient 21115 is the number of quarters which, with the remainder 21 pounds, makes up the whole stated quantity. On dividing this number of quarters by 4, we get the number of hundred-weights 5278 which, with the remainder 3 quarters, and the previous remainder 21 pounds, makes up the whole quantity. Lastly, on dividing the number of hundred-weights by 20, we get 263 as the number of tons which, with the remainder 18 cwt. and the previous remainders 3 qrs. and 21 ℔s., makes up the whole original quantity. That is to say, the original quantity 591241℔s. is reduced to 263 tons, 18 cwt. 3 qrs. 21 ℔s.

```
         ℔s.
28 {4)5 9 1 2 4 1
   {7)1 4 7 8 1 0   1
      4)2 1 1 1 5   2 1
      2,0)5 2 7,8   3  2 1
  Tons 2 6 3-1 8-3-2 1,
           answ.
```

```
    t.  c. q. ℔s.
   263 18  3  21
        20
      5278
         4
     21115
        28
    168941
    42230
℔s. 591241, proof.
```

Exercises.	*Answers.*
30. Reduce 137 tons to hundred-weights......	2740
31. ——— 47 c. 3 q. 24 ℔s. to pounds........	5372
32. ——— 135 c. 3 q. 11 ℔s. to pounds......	15215

	Exercises.	*Answers.*
33.	Reduce 35 c. 2 q. 19 lbs. to pounds	3995
34.	——— 313 c. 1 q. 25 lbs. to pounds	35109
35.	——— 1 ton to ounces	35840
36.	——— 214 c. 3 q. to pounds	24052
37.	——— 94 c. 1 q. 11 lbs. to pounds	10567
38.	——— 59 tons, 11 c. 8 lbs. to pounds	133400
39.	——— 57386 pounds to hundred-wts., &c.	512 cwt. 1 q. 14 lbs.
40.	——— 57386 quarters to tons, &c.	717 t. 6 cwt. 2 qrs.
41.	——— 1000000 pounds to tons, &c.	446 t. 8 c. 2 q. 8 lbs.
42.	——— 1000000 oz. to hundred-wts., &c.	558 cwt. 0 qr. 4 lbs.
43.	——— 40865 pounds to hundred-wts., &c.	364 cwt. 3 q. 13 lbs.
44.	——— 92950 pounds to tons, &c.	41 t. 9 c. 3 q. 18 lbs.

REDUCTION OF TROY WEIGHT.

Exam. 4. Reduce 1 lb. 5 oz. 12 dwts. 13 grains, to grains.

```
lb. oz. dwt. grs.
 1   5   12   13
12                      Grains.
17              24{6)8461
20                {4)1410   1
352                2,0)35,2  13
 24                 12)17  12 13
1421                  lb. 1-5-12-13
 704                         proof.
8461 grains, answ.
```

In this example the multipliers are 12, 20, and 24, because in each pound there are 12 ounces, in each ounce 20 penny-weights, in each penny-weight 24 grains; and the 5 ounces, 12 penny-weights, and 13 grains are added in, as in the former examples. In the proof the divisors are 24, 20, and 12; the factors 6 and 4 being used for the 24, and the true remainder left in the division by 24 being found by the method given in page 42.

Exam. 5. Reduce 111111 grains to pounds.

```
        Grains.
 24{6)111111
   {4) 18518  3
   2,0)462,9  15
    12)231   9  15
     lbs. 19-3-9-15
             answ.
```

In this example the number of grains is divided by 24 (or by 6×4), to find an expression in penny-weights and grains for the same weight; the quotient, 4629, being thus found as the number of penny-weights which, with the remainder 15 grains, makes up the original weight of 111111 grains. That remainder 15 is found by the method given in page 42. The number of penny-weights is then divided by 20, and the whole original weight is found to be equal to 231 oz. 9 dwts. 15 grs. Next the number of ounces is divided by 12, and the whole original weight is found to be equal to 19 lbs. 3 oz. 9 dwts. 15 grs. The operation would be proved in the manner in which the last example was wrought.

Exercises.	*Answers.*
45. Reduce 11 oz. 12 dwts. 12 grs. to grs....	5580
46. ——— 3 lbs. 7 oz. to penny-weights...	860
47. ——— 1785 grains to ounces, &c........	3 oz. 14 dwts. 9 grs.
48. ——— 29 pounds to grains..............	167040

REDUCTION OF LONG MEASURE.

Exam. 10. Reduce 53 miles, 3 furlongs, 12 perches, 4 yards, to yards.

In this example the earlier steps are obvious, but some explanation may be wanted respecting the reduction of the perches to yards. Since one perch contains $5\frac{1}{2}$ yards, or five yards and half a yard, 17092 perches must contain 17092 times 5 yards, and 17092 times a half-yard; or, in other words, that length must contain 5 times 17092 yards and one half of 17092 yards. These two numbers of yards are set down in the work as 85460 and 8546. Further, to find the sum total of yards in the originally given length, the 4 yards given are added in *mentally* with those two numbers of yards already found.* This example would be proved in the manner in which the next is wrought.

M.	F.	P.	Y.
53	3	12	4
8			
427 furlongs			
40			
17092 perches			
$5\frac{1}{2}$			
85460			
8546			
94010 yards, *answ.*			

Exam. 11. Reduce 231278 yards to miles, &c.

In this example the number of yards is multiplied by 2, to reduce the yards to half-yards, and the result is divided by 11, the number of half-yards in a perch, to reduce it to perches. The remainder is 6 half-yards, or 3 yards; and the answer is found to be 131 miles, 3 furlongs, 10 perches, and 3 yards. The proof would proceed in the same manner in which the last example was wrought, and the pupil should perform it. Had the remainder been 7 half-yards instead of 6, it would have been equivalent to 3 yards, and 1 foot, 6 inches.

	Yards.	
	2 3 1 2 7 8	
	2	
1 1)	4 6 2 5 5 6	half-yards
4,0)	4 2 0 5,0	6 half-yards, or 3 yds.
8)	1 0 5 1	10 perches
Miles	1 3 1	3 10 3, *answ.*

* The process of multiplication by $5\frac{1}{2}$ here required is a process in what is called "*multiplication by a mixed number*"—a process of which a few preliminary explanations have been given in an example at page 54, and in respect to which fuller explanations will be given in the more advanced chapters on fractional arithmetic.

Exercises.	*Answers.*
49. Reduce 51 feet 4 inches to inches......	616
50. ——— 3 miles 3 furlongs to yards ...	5940
51. ——— 94 miles, 1 fur. 6 per. to per..	30126
52. ——— 571 leagues 2 miles to miles...	1715
53. ——— 573911 yards to miles, &c.....	326m. 0f. 27p. 2y. 1f. 6i.
54. ——— 23456 feet to miles, &c.........	4m. 3f. 21p. 3y. 0f. 6i.
55. ——— 25 miles and 34 per. to feet...	132561
56. ——— 1000000 inches to miles, &c...	15m. 6f. 10p. 2y. 2f. 4i.
57. ——— 100 miles to inches.............	6336000

REDUCTION OF CLOTH MEASURE.

Exercises.	*Answers.*
58. Reduce 28 yds. 3 qrs. 2 nails to nails *......	462
59. ——— 5247 nails to English ells, &c.......	262 e. 1 q. 3 n.
60. ——— 58 yards to nails........................	928
61. ——— 285 nails to yards, &c.	17 y. 3 q. 1 n.
62. ——— 468 yards to English ells, &c........	374 e. 2 q.

REDUCTION OF SUPERFICIAL OR SQUARE MEASURE.

Exercises.	*Answers.*
63. Reduce 245 square perches to sq. inches....	9604980
64. ——— 1325419 square inches to sq. yds. &c.	1022 y. 6 f. 43 i.

REDUCTION OF LAND MEASURE.

Exam. 12. Reduce 37 acres, 3 roods, 12 perches, to square yards.

This example, while it is one in Reduction of Land Measure, is one which might be also put under the previous heading, Reduction of Superficial Measure. The information required for working it will be obtained partly in the Table of Land Measure and partly in the Table of Square Measure.†

```
 A.  R.  P.
 37  3  12
  4
 ---
151 roods
 40
 ----
6052 perches
 30¼
------
181560
  1513
------
183073 yards, answ.
```

* Multiply the number of the yards by 4 to get a number of quarters, and take in the given quarters, and multiply the resulting number by 4 to get a number of nails, and take in the given nails. To work the next exercise, divide the given number of nails by 4; the quotient is the number of quarters, which is to be divided by 5 to find ells in the answer.

† The pupil should guard against a rather frequent error of speaking of "square acres." We have square roods, square perches, square feet, &c., but as there is no lineal acre, we ought not to speak of a square acre, unless, indeed, we wish to refer to an acre in the form of a square.

Exam. 13. Reduce 111111 square yards to acres, &c.

For working this example it is to be noticed that 1 sq. perch contains $30\frac{1}{4}$ sq. yards, or 121 quarters of a square yard. On this account, to reduce the length given in sq. yards to perches, it is first reduced to quarter-yards, and then the number of quarter-yards is divided by 121 to get perches with quarter-yards remaining. Thus there is obtained 3673 perches and 11 quarter-yards; but 11 quarter-yards are $= 2\frac{3}{4}$ yards. The rest of the operation for finding the acres, &c., is obvious.

```
                 Yards.
                 111111
                      4
      ⎧ 11)444444 quarters of a yard.
  121 ⎨ 11)40404
      ⎩
       4,0)367,3 p. 11 q.yds. or 2¾ yds.
          4)91 r. 33 p. 2¾ yards.
   Answ.   22 a. 3 r. 33 p. 2¾ yds.
```

Exercises.	*Answers.*
65. Reduce 234 acres, 1 r. 13 per. to perches......	37493
66. ——— 93827 perches to acres, &c..............	586 a. 1 r. 27 p.
67. ——— 256 acres, 15 perches, to yards.........	$1239493\frac{3}{4}$

Besides the foregoing kinds of reduction of land measure, which have had for their object to reduce areas given in one or more denominations of one special "measure" to their expression in one or more other denominations of the same "measure," there is another kind of reduction, often of essential practical importance, which has for its object the following:—To reduce the expression of an area given in one special measure to the expression for the same area in a different special measure. (In these statements the name *special measure* is used to refer, for instance, to any of the following systems of land measure as distinct from the others:—English or Imperial Statute Measure, Irish Plantation Measure, Cunningham Measure, Scotch Measure, &c.) Questions of this kind are not easy enough to be well suited for ordinary pupils learning reduction for the first time, and the example which will be given here for the benefit of more advanced learners may well be omitted by beginners. This one example may suffice for suggesting how to proceed in other cases when necessary.

Exam. 14. Reduce 243 acres, 3 roods, 17 perches, Irish Plantation Measure, to English or Imperial Statute Measure.

Here we have first to notice, from the information given in a note on page 59, that 121 acres, Irish Plantation Measure, are=196 acres English or Imperial Statute Measure. It follows obviously that 1 acre, Irish, $=\frac{196}{121}$sts of the statute acre, and hence we may also see that 1 Irish perch $=\frac{196}{121}$sts of the statute perch; because an Irish acre is 4×40 Irish perches, and a statute acre is 4×40 statute perches. Now, to solve the question, a good way is to commence by reducing the area given in Irish acres, roods, and perches to Irish perches, so as to get it expressed in one denomination. Thus, as is shown in the margin, we find the number of Irish perches to be 39017. Now

```
  A.  R.  P.
 243  3  17
   4
 975
  40
39017 perches, Irish.
```

$$1 \text{ Irish perch} = \frac{196}{121}\text{sts of the statute perch;}$$

and therefore, 30917 Irish perches $=\dfrac{30917\times196}{121}$sts of the statute perch,

$$= 63201\tfrac{11}{121} \text{ statute perches.}$$

This is the required result stated in perches. The fraction $\frac{11}{121}$ may obviously be reduced to lower terms, since its numerator and denominator are each divisible by 11; and so we find that it is equivalent to $\frac{1}{11}$. Then, by ordinary reduction of $63201\frac{1}{11}$ perches to acres, &c., we find as the result in English or Imperial Statute Measure, 395 acres, 0 roods, $1\frac{1}{11}$ perch.

REDUCTION OF TIME.*

Exercises.	*Answers.*
68. Reduce 17 days to minutes	24480
69. ——— 1 day, 4 h. 12 sec. to seconds	100812
70. ——— 12345678 seconds to days, &c.	142d.21h.21m.18s.
71. ——— 11,000,010 seconds to days, &c.	127d.7h. 33m.30s.

Exam. 15. How many days are there from the 8th of May till the 23rd of July?†

31 days in May
8
23 May
30 June
23 July
76 days, *answ.*

Exer. 72. How many days are there from the 12th of August till the 24th of the next April in a common year? *Answ.* 255.

73. In a leap year, how many days are there from the 8th of January till the 12th of December? *Answ.* 339.

74. How many days are there between the 17th of March, and the 25th of December?‡ *Answ.* 283.

75. Reduce the earth's equatorial and polar diameters, and also their difference, and half their sum, to miles.—(*See Exercise* 75, *page* 46.) *Answ. Equatorial,* $7925\frac{3426}{5280}$ m.; *polar,* $7899\frac{900}{5280}$ m.; *difference,* $26\frac{2526}{5280}$ m.; *half sum,* $7912\frac{2163}{5280}$ m.

76. "In the year 1815, Catherine Woods, of Dunmore, near Ballynahinch, in the County of Down, then about thirteen years of age, spun a hank of linen yarn, of 12 *cuts*, each cut 120 threads, each thread two yards and a half, which weighed ten grains;" being at the rate of 700 hanks to the pound avoirdupois.§ It is required to determine the length into which, at this rate, a pound of flax would have been extended.—(*Stuart's "Historical Memoirs of the City of Armagh,"* p. 424.) *Answ.* 2,520,000 yards, or $1431\frac{144}{176}$ miles.

77. If one person were to lie in bed nine hours each day at an

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* Reductions of cubic measure, and of liquid and dry measure, are so easy, after what has gone before, that it is unnecessary to give exercises regarding them.

† This, and the similar questions which follow it, though not strictly of the same kind as the others in reduction, are inserted on account of their utility.

‡ Questions of the following kind are frequently useful:—

1. If the 8th of August be on Monday, on what day of the week will the 1st of November be?

The number of days between these dates is found to be 85, which, being divided by 7, becomes 12 weeks and 1 day; counting 1 day, therefore, after Monday, we find that the 1st of November must be on Tuesday.

2. If a common year begin on Friday, on what day will the 18th of June, the anniversary of the battle of Waterloo, happen? *Answ.* On Friday.

3. If, in a leap year, the 9th of July be on Friday, on what day did the year commence? *Answ.* On Thursday.

4. If a leap year commence on Wednesday, what day of September will be the first Monday of that month? *Answ.* The 7th.

Here, let the day be found on which September will commence, and the rest is easy.

§ For this extraordinary, and perhaps unequalled performance, a premium of fifteen guineas was awarded by the Linen Board of Ireland. $17\frac{1}{2}$ lbs. of such yarn would contain a thread more than equal to the circumference of the Earth. —*Stuart's Memoirs.*

average, and another only six hours and a half; and if the latter were to employ the time thus gained in useful occupations, for forty years; to how many years' work, of 12 hours each day, would the entire time gained be equivalent? *Answ.* 8 years, 121 days, 8 hours.

78. The moon revolves round the earth in 27 days, 7 hours, 43 minutes, 12 seconds; and the nearest of Jupiter's and Saturn's moons complete their revolutions in 1 day, 18 hours, 28 minutes, and 22 hours, 38 minutes, respectively. Reduce all these to seconds, and find how often each of the second and third results is contained in the first. *Answ.* 2,360,592 seconds; 152,880 seconds; and 81,480 seconds; quotients, $15\frac{67392}{152880}$, and $28\frac{79152}{81480}$.

79. In what time would a body move from the earth to the moon, at the rate of thirty-one miles per hour; the mean distance being 237,628 miles? *Answ.* 319 days, $9\frac{13}{31}$ hours.

80. In what time would a body, moving with the velocity of sound, pass from the earth to the sun; the distance being 91,600,000 miles, and $365\frac{1}{4}$ days being taken as the period to be called a year in the answer?*—(*See Exercise* 55, *page* 30.) *Answ.* 13 years, 205 days, 12 hours, 51 minutes, $19\frac{73}{113}$ seconds.

81. In England there are 50,535 square miles; and in Wales, 7425. Find the number of acres contained in both. *Answ.* 37,094,400.

82. The circumference of the orbit described each year by the earth, is about 576,000,000 miles. In how many years, of $365\frac{1}{4}$ days each, would this space be described by a railway train, moving at the rate of 36 miles an hour? *Answ.* 1825 years, 85 days, 10 hours.

83. Reduce 125° 32′ 47″ to seconds. *Answ.* 451967″.

84. Reduce 328 degrees, $13\frac{1}{3}$ minutes to right-angles, degrees, minutes, and seconds. *Answ.* 3 right-angles, 58 degrees, 13 minutes, 20 seconds.

85. Reduce one radian to degrees, minutes, and seconds, approximately. *Answ.* 57° 17′ 45″.

86. Reduce 167° 43′ to radians, with a fraction of a radian, approximately. *Answ.* $2\frac{3187}{3438}$ radians.†

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* In this exercise, and also in Exer. 82, where it is wanted to reduce an ascertained number of hours to years, days, and hours, the years being each taken as $365\frac{1}{4}$ days, a good way to proceed will be to find, first, not days and hours, but quarter-days and hours (if any hours there be additional to the quarter-days), and then to reduce the quarter-days to years and quarter-days by dividing by $4 \times 365\frac{1}{4}$, or 1461, the number of quarter-days in the period here called a year; and then having got the number of years, the next step will be to reduce the quarter-days to days and hours, and then to add these hours to the hours found previously (if any there were; though in one of the two exercises it will be found that there happen to be none).

† When the pupil has learned to work with decimal fractions, it will ordinarily be better to express a result of this kind decimally. In that way the answer here would be 2·927 radians.

COMPOUND ADDITION.*

RULE. (1.) Arrange the given quantities so that those in each column may be of the same denomination. (2.) Add the quantities of the lowest denomination together, and reduce to the next higher denomination any part of their sum that can be so reduced: set the remainder below the column added, and carry the reduced part to the next. (3.) Proceed thus with all the other denominations, except the highest, which is to be added in the same manner as numbers in simple addition.

Either of the first two methods of proof given in simple addition may be employed in compound addition.

Exam. 1.

£	s.	d.
39	18	$7\frac{3}{4}$
51	12	$4\frac{1}{2}$
79	19	$10\frac{1}{2}$
8	7	$11\frac{1}{4}$
43	13	$9\frac{3}{4}$
375	16	$10\frac{3}{4}$
£599	9	$6\frac{1}{2}$, *sum*.‡

In this example the sum of the farthings is 14, out of which we can take one penny, or four farthings, 3 times, leaving a remainder of 2 farthings, or a halfpenny, and that is set down. The 3 pence taken out is † then added with the pence; the sum is 54 pence, out of which, by dividing 54 by 12, we find we can take 4 shillings, leaving a remainder of 6 pence, which is set down. The 4 shillings is then added with the units column of shillings: the sum is 39, of which the latter figure is set down, and the 3 being carried to the tens column of the shillings, the sum 8 is obtained; which number being divided by 2 (because 2 tens of shillings, or 20 shillings, make a pound), shows 4 as a

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* For the definitions of the several compound rules, see the corresponding simple rules, pages 15, 20, 23, and 36.

† It is quite allowable here to say "the 3 pence *is* added with the pence;" we are not bound by grammar in such cases to say "the 3 pence *are*." Likewise, farther on in the same example we are at liberty, and we may prefer to say, "the 4 shillings *is* then added with the units column of shillings," rather than "the 4 shillings *are* added." In either case we may be considering a quantity or a sum of money rather than a number of separate coin pieces. A person might be asked, "What did you do with the seven pence I gave you?" and might very well say, "I paid it away in settling the milk account." That might be said more readily than "I paid *them* away." There might not have been seven separate penny-pieces at all. It might have been, for instance, a silver sixpenny-piece, and a halfpenny-piece, and two farthing-pieces. Again, in reference to larger sums of money, a person might say of some one else, "Besides other debts, he owed me twenty-five pounds as rent. He paid me that £25, and I lodged it in bank." In such cases people would not usually say of twenty-five pounds, "I lodged *them* in bank."

‡ The proof is left to exercise the learner, and it will be proper for him to prove, not only this example, but all the exercises in this rule.

number of pounds; and that is to be carried to the pounds, and added in with them as in simple addition.*

In this example, the halfpence amount to 5, or 2 pence halfpenny. In such examples, where all the fractional parts are halfpence, it is better to call each a halfpenny, than 2 farthings.

Exam. 2.

£	s.	d.
15	3	$8\frac{1}{2}$
31	16	$7\frac{1}{2}$
94	13	$8\frac{1}{2}$
55	12	$11\frac{1}{2}$
37	11	$9\frac{1}{2}$
£234	18	$9\frac{1}{2}$, *sum.*

When the columns are very long, the work becomes heavy and laborious. In such a case the given quantities may be separated into two or more divisions, as is suggested in page 17.

Exercises.

1.			2.			3.			4.		
£	s.	d.	£	s.	d.	£	s.	d.	£	s.	d.
485	12	$7\frac{3}{4}$	3	14	$8\frac{1}{2}$	413	13	$10\frac{1}{4}$	4	13	$6\frac{1}{2}$
49	16	$3\frac{1}{2}$	0	19	$7\frac{1}{2}$	1245	10	9	3	15	$7\frac{3}{4}$
186	13	$11\frac{3}{4}$	5	17	$9\frac{3}{4}$	7085	15	$11\frac{3}{4}$	7	10	11
787	10	$8\frac{1}{4}$	2	12	6	8519	6	$4\frac{1}{2}$	1	12	$9\frac{1}{4}$
239	9	$9\frac{1}{2}$	2	16	$10\frac{1}{2}$	3456	14	10	2	8	$7\frac{1}{2}$
843	11	$4\frac{1}{2}$	1	5	$8\frac{1}{4}$	90	12	$5\frac{1}{2}$	4	0	3
374	16	7	3	8	3	69	15	$2\frac{3}{4}$	6	16	$8\frac{3}{4}$
285	4	$9\frac{3}{4}$	5	2	$4\frac{1}{2}$	179	18	$11\frac{1}{2}$	7	6	2
599	19	8	10	14	$5\frac{1}{4}$	788	9	9	9	15	$10\frac{1}{2}$

5. Add together £59 - 12 - $7\frac{3}{4}$; £95 - 14 - $2\frac{1}{2}$; £345 - 5 - $9\frac{1}{4}$; £88 - 16 - $2\frac{3}{4}$; £186 - 17 - $4\frac{1}{2}$; £347 - 7 - 6; £3 - 2 - $9\frac{1}{2}$; £7 - 14 - $7\frac{3}{4}$; £52 - 8 - $6\frac{3}{4}$; £59 - 3 - 4; £42 - 18 - $10\frac{1}{2}$; £187 - 10 - $10\frac{1}{4}$; and £954 - 16 - $5\frac{1}{4}$.

6. Add together £324 - 14 - $10\frac{1}{2}$; £518 - 5 - $9\frac{1}{2}$; £39 - 15 - 6; £54 - 11 - $11\frac{1}{2}$; £49 - 1 - $8\frac{1}{2}$; £9 - 7 - $11\frac{1}{2}$; £1000; £86 - 6 - $3\frac{1}{2}$; £324; £79 - 11 - $6\frac{1}{2}$; £5 - 13 - 9; £611 - 4 - $2\frac{1}{2}$; £186 - 13 - $1\frac{1}{2}$; and £476 - 8 - 5.

7. Required the sum of £21 - 0 - $10\frac{1}{2}$; £73 - 18 - 9; £22 - 3 - $7\frac{1}{2}$; £64 - 16 - 9; £19 - 18 - $1\frac{1}{2}$; £78 - 9 - 9; £16 - 9 - $10\frac{1}{2}$; £250 - 9 - 6; £17 - 12 - $7\frac{1}{2}$; £54 - 0 - $7\frac{1}{2}$; £797 - 12 - 9; £1 - 14 - $1\frac{1}{2}$; and £60 - 5 - 9.

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* PENCE TABLE.

d.		s.	d.	d.		s.	d.	d.		s.	d.	d.		s.	d.	d.		s.	d.	d.		s.	d.
12	are	1	0	40	are	3	4	72	are	6	0	100	are	8	4	132	are	11	0	160	are	13	4
20	..	1	8	48	..	4	0	80	..	6	8	108	..	9	0	140	..	11	8	168	..	14	0
24	..	2	0	50	..	4	2	84	..	7	0	110	..	9	2	144	..	12	0	170	..	14	2
30	..	2	6	60	..	5	0	90	..	7	6	120	..	10	0	150	..	12	6	180	..	15	0
36	..	3	0	70	..	5	10	96	..	8	0	130	..	10	10	156	..	13	0	200	..	16	8

This table has been inserted, lest some teachers should consider the want of it an imperfection. It seems better, however, not to impose upon the learner the labour of committing it to memory, except, perhaps, a small part at the beginning. If, instead of using it, he divide the amounts of the pence in his operations by 12, he will soon acquire a readiness in this species of addition, which will be very valuable; and by practice he will soon know the table, without formally committing it.

8. Add together one thousand and six pounds, fifteen shillings, and three farthings; three hundred pounds, seventeen shillings, and a halfpenny; four thousand and ninety-six pounds, eight shillings, and three farthings; seven thousand pounds, eighteen shillings, and eleven pence; two pounds, and three halfpence; five hundred and eleven pounds, and sixpence halfpenny; eighty-six pounds; and five hundred and eight pounds, seven shillings.

9. Required the sum of £15 - 13 - 8½; £53 - 16 - 7¾; £199 - 6 - 3¾; £548 - 8; £458 - 0 - 8; £1 - 2 - 9; £2 - 5 - 6; £3 - 8 - 3; £1 - 14 - 1½; £2 - 16 - 10½; £3 - 19 - 7½; £180 - 18 - 8; and £31 - 13 - 4¾.

10. Add together 15*s.* 8½*d.*; 13*s.* 10¼*d.*; 16*s.* 6¾*d.*; 19*s.* 3*d.*; 16*s.* 7½*d.*; 5*s.* 8¼*d.*; 13*s.* 9*d.*; 12*s.* 7¼*d.*; 15*s.* 1¾*d.*; 11*s.* 4½*d.*; 9*s.* 1¼*d.*; 10*s.* 9½*d.*; 17*s.* 4¼*d.*; 18*s.* 4*d.*; 4*s.* 5½*d.*; and 3*s.* 10¾*d.*

11. £513 - 15 - 10¼ + £34 - 2 - 10½ + £183 - 17 - 3½ + £548 - 13 - 8¼ + £8 - 15 - 9½ + £88 - 15 - 9½ + £176 - 15 - 6¾ + £79 - 13 - 11 + £5 - 0 - 6½ + £189 - 13 - 8¼ + £195 - 8 - 10¾ + £179 - 16 - 6.

12. Add together £18 - 14 - 8½; £12 - 13 - 9½; £21 - 12 - 10; £32 - 9 - 10½; £63 - 13 - 9½; £16 - 4 - 8½; £35 - 14 - 9½; £17 - 16 - 7½; £23 - 15 - 9¼; £35 - 17 - 2½; £8 - 19 - 8; £12 - 10 - 0½; and £13 - 8 - 8½.

13. Find the sum of one hundred and two pounds, and ten pence; fifty-eight pounds, and three farthings; forty pounds, seventeen shillings, and five pence halfpenny; nine pounds, fifteen shillings, and eleven pence; seventy-three pounds, twelve shillings; thirty-nine pounds, fourteen shillings, and a farthing; fifty-nine pounds; eighty-two pounds, and nine pence halfpenny; and fifty pounds.

14. £918 - 12 - 7¼ + £51 - 16 - 8½ + £519 - 14 - 4½ + 83 - 16 - 10½ + £55 - 12 - 8¾ + £183 - 7 - 9 + £188 - 18 - 2¼ + £375 - 12 - 3½ + £491 - 11 - 4½ + £318 - 15 - 9¾ + £780 - 10 - 4 + £10 - 19 - 9¾ + £508 - 16 - 9.

15. £459 - 16 - 11 + £583 - 11 - 11¼ + £15 - 14 - 8½ + £191 - 14 - 8¾ + £99 - 13 - 6½ + £195 - 12 - 6½ + £17 - 18 - 4¾ + £473 - 18 - 0½ + £18 - 16 - 7½ + £5 - 8 - 9½ + £31 - 12 - 2½ + £63 - 3 - 11 + £5 - 11 - 8.

16. Add together £3 - 4 - 7¾; £41 - 12 - 4; £186 - 17 - 9½; £3 - 8 - 10½; £67 - 18 - 9½; £112 - 16 - 6¾; £73 - 15 - 5¼; £139 - 7 - 4½; £510 - 6 - 7½; £706 - 7 - 7½; £581 - 4 - 4¾; £608 - 19 - 6½; and £519 - 12 - 8¼.

17. Required the sum of £13 - 6 - 8; £88 - 14 - 7½; £175 - 16 - 4½; £1245 - 13 - 5¾; £512 - 16 - 6; £91 - 12; £78 - 14 - 5; £475 - 19 - 9¼; £839 - 16 - 6¾; £512 - 11 - 11½; £447 - 16 - 0¾; £43; and £74 - 7 - 11.

Exam. 3.

tons.	*cwt.*	*q.*	*lbs.*	
35	16	0	20	
42	14	2	18	
18	9	1	16	
17	18	3	7	
31	5	3	19	
45	12	2	5	
191	17	2	1,	*sum.*

In this example, the numbers found for the sums of the pounds, quarters, and hundredweights, are respectively divided by 28, 4, and 20, the remainders set below their respective columns, and the quotients carried to the next columns, respectively. The hundredweights may be added, and the tons, if

any in them, may be taken out and carried forward in the same way as has been taught in Exam. 1, for adding shillings and carrying pounds forward from them; the divisor being 20 in both cases.

Exer. 18.			19.			20.			21.	
cwt.	*qrs.*	*lbs.*	*lbs.*	*oz.*	*drs.*	*tons.*	*cwt.*	*qrs.*	*lbs.*	*oz.*
53	2	12	16	12	13	75	13	1	7	11
17	1	15	5	15	3	83	17	2	53	14
16	3	19	12	12	5	17	8	0	47	10
19	0	18	3	11	9	16	16	1	86	9
25	3	18	19	1	11	61	15	2	94	7
48	3	6	14	4	8	39	9	3	29	3
42	2	7	24	7	14	88	7	3	42	10

22. Add together 55 c. 3 q. 18 lbs.; 34 c. 2 q. 22 lbs.; 63 c. 1 q. 23 lbs.; 71 c. 0 q. 19 lbs.; 16 c. 3 q. 20 lbs.; 3 c. 3 q. 26 lbs.; 27 c. 2 q. 23 lbs.; 41 c. 3 q. 9 lbs.; 35 c. 1 q. 18 lbs.; 43 c. 2 q. 24 lbs.; 95 c. 0 q. 10 lbs.; 29 c. 2 q. 17 lbs.; and 32 c. 2 q.

23. Add together 17 c. 2 q. 17 lbs.; 22 c. 1 q. 27 lbs.; 55 c. 3 q. 19 lbs; 13 c. 1 q. 15 lbs.; 73 c. 2 q. 13 lbs.; 88 c. 2 q.; 48 c. 0 q. 23 lbs.; 32 c. 3 q. 19 lbs.; 52 c. 2 q. 22 lbs.; 43 c. 3 q. 20 lbs.; 53 c. 1 q. 18 lbs.; 31 c. 2 q. 26 lbs.; and 45 c. 1 q. 14 lbs.

24. Find the sum of 58 tons, 12 c. 3 q. 21 lbs.; 32 t. 11 c. 2 q. 20 lbs.; 19 t. 15 c. 1 q. 12 lbs.; 17 t. 17 c. 0 q. 17 lbs.; 5 t. 3 c. 1 q. 25 lbs.; 73 t. 15 c. 1 q. 12 lbs.; and 98 t. 16 c. 2 q. 22 lbs.

25. Add together 13 acres, 3 roods, 27 perches; 45 a. 1 r. 27 p.; 63 a. 2 r. 17 p.; 26 a. 2 r. 26 p.; 16 a. 3 r. 34 p.; 21 a. 0 r. 8 p.; 55 a. 2 r. 31 p.; 37 a. 2 r. 18 p.; 44 a. 2 r. 20 p.; 57 a. 0 r. 19 p.; 61 a. 3 r. 18 p.; 39 a. 2 r.; and 5 a. 1 r. 30 p.*

26. Find the sum of 19 a. 2 r. 28 p.; 10 a. 0 r. 20 p.; 11 a. 3 r. 16 p.; 19 a. 2 r. 18 p.; 14 a. 2 r. 25 p.; 17 a. 3 r. 22 p.; 15 a. 2 r. 19 p.; 4 a. 3 r. 27 p.; 24 a. 2 r. 24 p.; 149 a. 1 r. 14 p.; 719 a. 2 r. 16 p.; and 107 a. 1 r. 23 p.

27. Add together 3 feet, 6 inches; 17 ft. $4\frac{1}{2}$ ins.; 4 ft.; 3 ft. $3\frac{3}{4}$ ins.; 2 ft. 4 ins.; 7 ft. 11 ins.; $5\frac{3}{4}$ ins.; 1 ft. $2\frac{1}{4}$ ins.; 2 ft. $0\frac{3}{4}$ ins.

28. Find the sum of the following angles:—45° 6′ 20″; 30° 45′; 21° 19′ 40″; 90°; 6° 4′ 40″; 7° 9′; and 19° 7′ 20″.†

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* The units of perches may be added as in simple addition, and the sum of the tens divided by 4, the tens in 40. In like manner, the units of minutes and seconds may be added as in simple addition, and the sum of the tens divided by 6, the tens in 60.

† The examples and exercises given above in compound addition are judged to be sufficient. They include the cases most commonly requisite for practical use; and a pupil, having understood and practised these, may readily perform compound addition in other denominations, or in other kinds of measured things, when requisite, by the application of the same principles on which these depend.

COMPOUND SUBTRACTION.

Rule. (1.) Place the less quantity below the greater, arranging that the quantities in each column may be of the same denomination. (2.) Then, beginning with the lowest denomination, subtract, if possible, each number in the lower line from that which stands above it. (3.) But when in any denomination this cannot be done, take the number in the lower line from the number which, in that denomination, is equivalent to a unit in the next higher; to the remainder add the upper number for the remainder to be set down in that denomination, and carry one to the number in the next higher denomination in the lower line. (4.) Proceed thus with all the denominations except the highest, in which the work is to be performed as in simple subtraction.

The *methods of proof*, and the *principles* on which the operations depend, are the same as in simple subtraction.

Exam. 1. From £33 - 17 - $10\frac{3}{4}$, take £18 - 8 - $4\frac{1}{4}$.

£	s.	d.
33	17	$10\frac{3}{4}$
18	8	$4\frac{1}{4}$
£15	9	$6\frac{1}{2}$, *answ.*

Exam. 2. Required the difference between £159 - 9 - $4\frac{1}{4}$ and £86 - 17 - $8\frac{1}{2}$.

In this example, as a halfpenny is greater than a farthing, it is taken from a penny, and the remainder being added to the farthing, the sum, 3 farthings, is set down; a penny is then carried to 8 pence, and the sum being taken from one shilling or 12 pence, the remainder 3 is added to the 4 pence, and the amount, 7 pence, is set down. We then proceed thus: carry 1, and 7 are 8; 8 from 9 and 1 remains, which we set down; then 1 from 2 (the tens' figure in the number of shillings in a pound), and 1 remains, and we set it down; carry 1, and 6 are 7; 7 from 9 and 2 remain, &c.

£	s.	d.
159	9	$4\frac{1}{4}$
86	17	$8\frac{1}{2}$
£72	11	$7\frac{3}{4}$, *answ.*

Exercises.

		£	s.	d.		£	s.	d.
1.	From	19	3	10	take	8	15	$3\frac{3}{4}$
2.	......	575	15	$1\frac{1}{2}$	......	124	13	4
3.	......	192	11	$4\frac{1}{2}$	......	88	16	$9\frac{1}{2}$
4.	......	511	3	$2\frac{1}{4}$	......	247	10	$6\frac{1}{2}$
5.	......	12	4	9	......	5	2	$4\frac{1}{2}$
6.	......	100	0	0	......	1	2	9

7. From £3 subtract 7 pence.
8. From £3 subtract 7 shillings.

		cwt.	*q.*	*lbs.*		*cwt.*	*q.*	*lbs.*
9.	From	13	3	20	take	9	2	12
10.	......	23	1	5	......	17	3	22
11.	......	105	0	0	......	79	1	13

	tons.	*cwt.*	*q.*	*lbs.*		*tons.*	*cwt.*	*q.*	*lbs.*
12. From	15	7	0	24	take	5	12	1	10

13. A person has to pay 8*s.* 2*d.* He has, in his purse, a half-crown, two florins, a sixpence, a fourpenny piece, and a threepenny piece. He has also some copper pence. Paying all the silver money away from his purse, how much must he add in coppers to make up the required payment?

14. A person has to pay £2 - 14 - 7, and he has with him just two sovereigns, a crown piece, three half-crowns, a florin, and a shilling piece. Giving all this in payment, how much change ought e to get back?

15. The latitude of Rome (St. Peter's) is 41° 53′ 54″ North; of Paris (Observatory of the Military School), 48° 51′ 6″ N.; of London (St. Paul's), 51° 30′ 49″ N.; of Dublin (Observatory), 53° 23′ 13″ N.; of Edinburgh (Observatory), 55° 57′ 57″ N.; and of St. Petersburgh, 59° 56′ 23″ N. Required the difference of the latitudes of the first and second, the second and third, &c., of these capitals.

16. The three angles of a triangle are always equal to two right angles. A land surveyor has measured two angles and found them to be 43° 21′, and 78° 54′. Find from these data what the third angle must be.

17. The latitude of the Cape of Good Hope is 33° 55′ 15″ S., and that of Cape Horn 55° 58′ 30″ S. Required their difference.

18. The latitudes of Belfast and Glasgow, are 54° 36′ N., and 55° 52′ N. respectively. Required their difference.

19. The following are the times in which the principal planets perform their revolutions round the sun; required the differences of the first and second, of the second and third, &c.:—

	Days.	*h.*	*m.*		*Days.*	*h.*	*m.*
Mercury.........	87	23	16	Jupiter	4332	14	2
Venus............	224	16	49	Saturn	10759	5	17
Earth	365	6	9	Uranus.........	30686	19	42
Mars	686	23	31				

20. Reduce the difference between the Julian year of 365 days, 6 hours, and the true year of 365d. 5h. 48m. 50s. to seconds: reduce also one day to seconds; and divide the second result by the first. *Answ.* 670 seconds, and 86400 seconds; quotient, $128\frac{64}{67}$.*

* This exercise explains the *old* and *new styles* in the reckoning of time; showing that by using the Julian year, that is, by taking every fourth year as a leap year, an error of a day is accumulated in about 129 years, or a little more than three days in 400 years. Hence, in every four centuries, three years, that, by the Julian reckoning, would be leap years, are taken as common years.—(*See pages* 60 and 61.)

21. A person goes to the sea-side for three days, and on leaving home he has with him £4 - 9 - 7. On returning home he finds he has noted the following expenses paid: — Cab, 1*s.* 6*d.*; porter, 4*d.*; railway fare, 3*s.* 7*d.*; newspaper, 1*d.*; time table, 4*d.*; boat hire, 7*s.* 6*d.*; fishing-hooks, $7\frac{1}{2}$*d.*: biscuits, 5*d.*; hotel bill, £1 - 15 - 4; railway fare, 3*s.* 7*d.*; cab, 1*s.*: and, on counting the money he has remaining, he finds it to be £1 - 14 - $4\frac{1}{2}$. Find whether his notes account for all the money spent; and, if not, by how much they are deficient.

COMPOUND MULTIPLICATION.

RULE I. *To multiply a quantity expressed in more denominations than one, by a number not exceeding* 12: * Commencing with the lowest denomination, multiply, successively, the several numbers in the multiplicand by the multiplier, dividing, setting down, and carrying, as in compound addition.

Exam. 1. Multiply £1 - 14 - $7\frac{3}{4}$ by 9.

£	s.	d.
1	14	$7\frac{3}{4}$
		9
£15	11	$9\frac{3}{4}$, *answ.*

In this example, the farthings, pence, and shillings are multiplied successively by 9; the numerical products,† as they are found, are respectively divided by 4, 12, and 20 (or the tens of the shillings by 2); the several remainders are written down, and the quotients carried. The pounds are multiplied as in simple multiplication; and the product is found to be £15 - 11 - $9\frac{3}{4}$.

Exercises.

	£	s.	d.			£	s.	d.			£	s.	d.	
1.	3	14	$9\frac{1}{2}$	$\times 2$	5.	3	15	$10\frac{1}{4}$	$\times 6$	9.	0	15	$10\frac{1}{4}$	$\times 10$
2.	1	17	$8\frac{3}{4}$	$\times 3$	6.	1	2	9	$\times 7$	10.	9	7	8	$\times 11$
3.	0	18	$11\frac{1}{4}$	$\times 4$	7.	2	18	9	$\times 8$	11.	3	19	$7\frac{1}{2}$	$\times 12$
4.	1	10	4	$\times 5$	8.	6	13	4	$\times 9$	12.	1	16	7	$\times 12$

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* Or *not exceeding* 19, if the learner have committed to memory the supplement to the multiplication table, page 25. The same circumstance will also modify the succeeding problems in a similar manner.

† The reason for saying here, not that the products are divided, but that the *numerical products* are divided by 4, 12, and 20, is, that when we multiply the three farthings by 9, we get *twenty-seven farthings* as the product. Now, it is not this *quantity of money* that we are to divide by 4, but the *number* expressing it in farthings, viz. 27, that we are to divide by 4, to get a number of pence with a remainder of some farthings. Thus we get 6 pence with 3 farthings. If we were to divide the quantity, *twenty-seven farthings*, by 4, that is, to take a fourth part of it, we would get 6 farthings and a fourth of three farthings, for which, however, it happens we have no coin small enough. Like explanations might be given in respect to the pence and shillings.

RULE II. *To multiply by a number which exceeds* 12, *but is the product of two or more factors, each less than* 13; (1.) By the preceding rule, multiply the given multiplicand by one of the factors. (2.) Multiply the result by another. (3.) Multiply this last result by another, if there be so many; and thus proceed, whatever is their number.

Exam. 2. Multiply 18*s.* $3\frac{1}{4}$*d.* by 42.

In this example, the multiplicand is multiplied by 6, and the product is £5 - 9 - $7\frac{1}{2}$. This again is multiplied by 7, and the product is £38 - 7 - $4\frac{1}{2}$. The *reason* of the operation is sufficiently obvious, since 42 is the product of 6 and 7. The work might be proved by multiplying the multiplicand by 7, and the result by 6.

£	s.	d.
0	18	$3\frac{1}{4}$
		6
5	9	$7\frac{1}{2}$
		7
£38	7	$4\frac{1}{2}$, *answ.*

When the multiplicand contains one or more farthings, if one of the factors be *even*, it is better to use it first, as the farthings may thus disappear, in which case the rest of the work will be easier. But if the multiplicand end in pence, without farthings, and one of the factors be 3, 6, or 9, it is better to use that factor first, as the pence may thus disappear; and, in all cases in multiplication of money, when 12 is one of the factors, it should be used first, as part of the operation may be performed by inspection, by setting down 3 pence for each farthing, and carrying to the shillings 1 for every penny in the multiplicand. Thus, in multiplying £1 - 13 - $7\frac{1}{4}$ by 12, set down 3 pence, and carry 7 to the shillings, saying, 12 times 3 are 36, and 7 are 43, &c.

£	s.	d.
1	13	$7\frac{1}{4}$
		12
£20	3	3, *answ.*

	Exercises. £	s.	d.				*Answers.* £	s.	d.
13.	1	17	$9\frac{1}{2}$	×	14	....	26	9	1
14.	0	13	$4\frac{1}{2}$	×	15	....	10	0	$7\frac{1}{2}$
15.	0	2	$8\frac{1}{2}$	×	16	....	2	3	4
16.	3	14	$0\frac{1}{4}$	×	18	....	66	12	$4\frac{1}{2}$
17.	1	5	3	×	21	....	26	10	3
18.	0	14	1	×	22	....	15	9	10
19.	0	13	2	×	24	....	15	16	0
20.	0	12	$7\frac{1}{4}$	×	27	....	17	0	$3\frac{3}{4}$
21.	2	15	5	×	32	....	88	13	4
22.	1	16	$6\frac{1}{4}$	×	33	....	60	5	$2\frac{1}{4}$
23.	1	19	4	×	36	....	70	16	0
24.	1	11	8	×	42	....	65	12	6
25.	0	8	$4\frac{1}{2}$	×	45	....	18	16	$10\frac{1}{2}$
26.	2	12	$10\frac{1}{2}$	×	48	....	126	18	0
27.	0	16	$4\frac{3}{4}$	×	49	....	40	3	$4\frac{3}{4}$
28.	0	3	$10\frac{1}{4}$	×	56	....	10	15	10
29.	0	6	$11\frac{1}{4}$	×	60	....	20	16	3
30.	2	10	5	×	63	....	158	16	3

	Exercises.						*Answers.*		
	£	s.	d.				£	s.	d.
31.	1	14	4	×	66	....	113	6	0
32.	0	4	$5\frac{1}{4}$	×	72	....	15	19	6
33.	1	6	$9\frac{3}{4}$	×	80	....	107	5	0
34.	1	6	7	×	84	....	111	13	0
35.	1	10	8	×	90	....	138	0	0
36.	1	15	$11\frac{3}{4}$	×	96	....	172	14	0
37.	1	17	2	×	100	....	185	16	8
38.	0	16	$11\frac{1}{2}$	×	108	....	91	11	6
39.	0	18	9	×	110	....	103	2	6
40.	0	4	$8\frac{1}{4}$	×	120	....	28	2	6
41.	2	19	6	×	121	....	359	19	6
42.	1	16	$7\frac{1}{2}$	×	132	....	241	14	6
43.	2	19	$2\frac{1}{4}$	×	144	....	426	3	0
44.*	1	12	5	×	75	....	121	11	3
45.	1	18	$8\frac{3}{4}$	×	112	....	216	17	8
46.	0	14	$2\frac{3}{4}$	×	128	....	91	1	4
47.	1	2	$7\frac{1}{4}$	×	147	....	166	2	$9\frac{3}{4}$
48.	0	11	$4\frac{1}{2}$	×	168	....	95	11	0
49.	1	16	8	×	175	....	320	16	8
50.	2	15	$2\frac{1}{4}$	×	196	....	540	16	9

RULE III. *To multiply by a number which exceeds* 12, *but is not produced by factors below* 13: (1.) Use, as in the last rule, those factors whose product is nearly equal to the multiplier. (2.) Increase or diminish the result, as the case may require, by the product of the multiplicand and the difference between the multiplier and the product of the factors employed.

Exam. 3. Multiply 14*s*. $10\frac{1}{2}$*d*. by 38.

In this example, 38 not being the product of any two factors not exceeding 12, we multiply by 36, as before, and to the product we add twice the multiplicand, to find the product by 38. The answer would have been obtained with nearly the same facility, had we multiplied by 40 (4 × 10), and subtracted twice the multiplicand; and thus the operation might be proved.

£	s.	d.	
0	14	$10\frac{1}{2}$	
		12	
8	18	6,	prod. by 12
		3	
26	15	6,	prod. by 36
1	9	9,	prod. by 2
£28	5	3,	prod. by 38

* In this and each of the six following exercises the multiplier is the product of *three* factors; since 75=3×5×5; 112=8×2×7, or 4×4×7; 128=8×8×2, or 4×4×8, &c. In like manner, 140=2×7×10; 192=12×4×4; 240=12×2×10; 1728=12×12×12, &c.

Exam. 4. What is the cost of 61 cwt. at £1 - 4 - 10 per cwt.?

The operation in this example is contracted by taking in the multiplicand, in the multiplication by 5. This contraction may always be employed, when the product of the factors is *one* less than the multiplier. A similar contraction might be employed, though not with the same facility, when the excess is 2, 3, &c.

£	s.	d.	
1	4	10	
		12	
14	18	0,	prod. by 2
		5	
£75	14	10,	prod. by 61

Exam. 5. Multiply £2 - 7 - 8½ by 79.

In this example, it is better to find the product by 80, and subtract the multiplicand from it, than to multiply by 77, and add twice the multiplicand to the product, as in the one way there is one multiplication fewer than in the other. A like contraction may be applied with advantage in every case in which factors can be found, whose product is *one* greater than the given multiplier, while factors cannot be found whose product is only one less than it.

£	s.	d.	
2	7	8½	
		8	
19	1	8,	prod. by 8
		10	
190	16	8,	prod. by 80
2	7	8½	to be subtr.
£188	8	11½,	prod. by 79

	Exercises.					*Answers.*		
	£	s.	d.			£	s.	d.
51.	5	3	2	× 13*	....	67	1	2
52.	2	8	7¾	× 23	....	55	18	10¼
53.	1	10	1	× 29	....	43	12	5
54.	0	17	3½	× 31	....	26	16	0½
55.	0	1	4½	× 38	....	2	12	3
56.	0	11	3¾	× 39	....	22	1	2¼
57.	0	18	8	× 46	....	42	18	8
58.	1	11	0¼	× 47	....	72	17	11¾
59.	2	6	5	× 52	....	120	13	8
60.	0	13	0¼	× 53	....	34	10	1¼
61.	2	5	4	× 58	....	131	9	4

* In multiplying by 13, if the pupil have committed the multiplication table no farther than 12 times 12, he may multiply by 12, and take in the ultiplicand as he proceeds, as in the annexed example. In this way the multiplication by any *multiple* of 13, not exceeding 13 times 13, may be performed in two lines, as also by any number which is greater by unity than any of these multiples. Hence, only two lines are necessary in multiplying by 26, 39, 52, 53, 65, 78, 79, 91, 92, 104, 105, 117, 118, 130, 131, 143, 156, 157, 169, or 170. (By any *multiple* of 13 is meant any number made up by the repetition of 13 any number of times; and here, in the phrase "repetition of 13 any *number* of times," the word "number" is to be understood as being restricted to its only proper meaning, so as not to include what are often called *fractional numbers*, but may be better called *fractional numerics*. Fuller information respecting the name multiple will be found farther on, in the chapter on measures and multiples.)

£	s.	d.	
1	15	3¾ × 13	
		12	
£22	19	0¾,	*answ.*

	Exercises.		*Answers.*
	£ s. d.		£ s. d.
62.	1 12 $1\frac{1}{4}$ × 65	....	104 6 $9\frac{1}{4}$
63.	0 19 $9\frac{3}{4}$ × 68	....	67 7 3
64.	0 12 $8\frac{1}{2}$ × 69	....	43 16 $10\frac{1}{2}$
65.	0 2 7 × 75	....	9 13 9
66.	3 7 11 × 76	....	258 1 8
67.	1 0 $9\frac{1}{2}$ × 78	....	81 1 9
68.	0 14 5 × 82	....	59 2 2
69.	0 6 8 × 87	....	29 0 0
70.	1 13 6 × 89	....	149 1 6
71.	1 3 $2\frac{3}{4}$ × 91	....	105 13 $10\frac{1}{4}$
72.	1 16 4 × 94	....	170 15 4
73.	2 2 $11\frac{1}{4}$ × 95	....	203 19 $0\frac{3}{4}$
74.	3 9 7 × 98	....	340 19 2
75.	0 15 $6\frac{3}{4}$ × 103	....	80 2 $11\frac{1}{4}$
76.	0 11 $4\frac{1}{2}$ × 104	....	59 3 0
77.	1 10 $6\frac{1}{4}$ × 105	....	160 4 $8\frac{1}{4}$
78.	2 5 10 × 111	....	254 7 6
79.	0 0 $9\frac{1}{4}$ × 113	....	4 7 $1\frac{1}{4}$
80.	3 15 $3\frac{3}{4}$ × 117	....	440 11 $6\frac{3}{4}$
81.	3 17 $8\frac{3}{4}$ × 122	....	474 2 $11\frac{1}{2}$
82.	2 13 $1\frac{1}{4}$ × 125	....	331 18 $0\frac{1}{4}$
83.	1 10 8 × 130	....	199 6 8
84.	0 6 $9\frac{1}{2}$ × 139	....	47 4 $0\frac{1}{2}$
85.	0 17 $4\frac{3}{4}$ × 145	....	126 2 $4\frac{3}{4}$
86.	0 15 7 × 150	....	116 17 6
87.	2 4 $8\frac{1}{2}$ × 155	....	346 9 $9\frac{1}{2}$
88.	2 11 $5\frac{1}{4}$ × 156*	....	401 4 3

* The subjoined example will show how compound multiplication may be performed, however great the multiplier is. This method, however, will scarcely be employed by any one who has learned the modes of computation usually taught in the rule of *practice*. It is therefore given in a note, and not in the text.

Exam. What cost 2485 yards of broadcloth, at 15*s*. $7\frac{1}{2}$*d*. per yard?

In this example we find successively the prices of 10, 100, and 1000 yards. We then multiply the price of 1000 by 2; of 100 by 4; of 10 by 8; and of 1 by 5. We have thus the prices of 2000, of 400, of 80, and of 5, the sum of which is £1941 - 8 - $1\frac{1}{2}$, the answer.

£	s.	d.		
0	15	$7\frac{1}{2}$	price of	1 yard
		10		
7	16	3	price of	10 yards
		10		
78	2	6	......	100 ....
		10		
781	5	0	......	1000 ....
		2		
1562	10	0	......	2000 ....
312	10	0	......	400 ....
62	10	0	......	80 ....
3	18	$1\frac{1}{2}$	......	5 ....
£1941	8	$1\frac{1}{2}$	......	2485 ....

[*This note is continued on next page.*]

Exam. 6. Multiply 33 cwt. 3 q. 22 lbs. by 7.*

In this example, as the operation proceeds, the number of pounds is divided by 28, and the number of quarters by 4.

cwt.	q.	lbs.	
33	3	22	
		7	
237	2	14,	*answ.*

	Exercises.		*Answers.*
	cwt. q. lbs.		*cwt. q. lbs.*
89.	1 2 17 × 27	....	44 2 11
90.	0 3 22 × 86	....	81 1 16

91. Required the cost of a chest of tea, containing 97 lbs., at 3*s.* $4\frac{1}{2}$*d.* per lb. *Answ.* £16 - 7 - $4\frac{1}{2}$.

92. Required the amount of a box of linen cloth, containing as under, and at the given rates per yard (in this, the two pieces first stated contain jointly 49 yards, at 2*s.* 1*d.* per yard; and so on with the others):—

piec.	*yd.*		*piec.*	*yd.*		*piec.*	*yd.*	
2	49	at 2*s.* 1*d.*	3	75	at 2*s.* 11*d.*	3	75	at 3*s.* 7*d.*
2	49	2 3	3	75	3 1	3	75	3 9
2	48	2 5	3	75	3 3	3	68	3 11
3	73	2 7	3	75	3 5	3	75	4 2
3	75	2 9				*Answ.*	£140 - 2 - 0.	

93. Required the cost of a hogshead of rum (in bond, duty not being paid) containing 61 gallons, at 5*s.* 4*d.* per gallon, and a puncheon containing 104 gallons, at 5*s.* 1*d.* per gallon. *Answ.* £16 - 5 - 4, and £26 - 8 - 8.

94. What cost a hundred-weight of indigo, at 6*s.* $4\frac{1}{2}$*d.* per pound? *Answ.* £35 - 14 - 0.

95. If a labourer receive 18*s.* 4*d.* per week, how much will he receive in 52 weeks of work? *Answ.* £47 - 13 - 4.

96. What is the amount of the duty on 100 gallons of brandy, at 10*s.* 5*d.* per gallon? *Answ.* £52 - 1 - 8.

97. What is the duty on 63 gallons of Jamaica rum, at 10*s.* 2*d.* per gallon? *Answ.* £32 - 0 - 6.

The following exercises may be wrought in a similar manner:—

	Exercises.		*Answers.*
	£ *s.* *d.*		£ *s.* *d.*
1.	3 6 $5\frac{1}{4}$ × 3178	....	10556 18 $4\frac{1}{2}$
2.	1 11 $3\frac{1}{4}$ × 15934	....	24913 9 $5\frac{1}{2}$
3.	2 6 $9\frac{1}{2}$ × 938	....	2194 10 7
4.	0 17 $1\frac{3}{4}$ × 63491	....	54430 6 $1\frac{1}{4}$

* Compound multiplication is seldom employed, except in relation to money; but if it should be necessary to use it in cases not illustrated here, no difficulty can arise, as the method is similar in all cases.

In the note on page 23 it has been explained that it would be absurd to speak of multiplying money by money, and it may be useful here to repeat the caution, the more so because much misapprehension has too frequently been exhibited on this subject. When we multiply a quantity by a proper number, we simply find what would be produced by repeating that quantity the given number of times: thus, if 2*s.* 6*d.* be repeated 4 times, the amount is 10*s.*; if 5 times, it is 12*s.* 6*d.*, &c. When we do what is called multiplying a quantity by a "fractional number," or rather by a fractional numeric, we just take that numeric of it: thus, when we do what is called multiplying 2*s.* 6*d.* by $2\frac{3}{5}$, we just take $2\frac{3}{5}$ths of 2*s.* 6*d.*; or we take 2 of 2*s.* 6*d.*, that is 5*s.*, and $\frac{3}{5}$ of 2*s.* 6*d.*, that is 1*s.* 6*d.*, making in all 6*s.* 6*d.*; or further, when we do what is called multiplying 2*s.* 6*d.* by $\frac{2}{3}$, we just take $\frac{2}{3}$ of it, and so we find 20 pence as the product. To talk, however, of multiplying 2*s.* 6*d.* by 2*s.* 6*d.*, or of what might as well be called taking 2*s.* 6*d.* of 2*s.* 6*d.*, is absurd. When the pupil shall have commenced to work in the rule of proportion, indeed, he may find that we sometimes *appear* to multiply quantities by quantities; as, for instance, to multiply money by money. In all such cases, however, on proper consideration, it will be found that it is not by a quantity, but by either a number or a fractional numeric, that we multiply.

We see from the nature of division that there is no absurdity in *dividing* money by money; that is, in finding how often one sum is contained in another, or how often one sum can be taken out of another, and this will be exemplified in the next chapter.

COMPOUND DIVISION.

RULE I. *To divide a quantity expressed in more denominations than one, by a number not exceeding* 12: (1.) Divide the quantity in the highest denomination by the given divisor, by short division. (2.) Reduce the remainder, if there be any, to the denomination next lower, and add to the result what was given of that denomination. (3.) Divide the sum by the divisor; and thus proceed to the lowest denomination, or till nothing remains.

Exam. 1. Divide £114 - 16 - $7\frac{1}{4}$ by 10.

In this example we divide the £114 by 10, and so we find £11 to be set down, and we have remaining £4, or 80 shillings; which, increased by the 16 shillings, becomes 96 shillings. We next divide this 96 shillings by 10, and so we find 9 shillings to be set down, and we have remaining 6 shillings, or 72 pence; which, increased by the 7 pence, becomes 79 pence. Dividing this by 10, we find 7 pence to be set down, and we have remaining 9 pence, or 36 farthings, which with the one farthing makes 37 farthings. Dividing this by 10, we get 3 farthings to be set down as $\frac{3}{4}$ of a penny; and we have 7

	£	*s.*	*d.*
10)	114	16	$7\frac{1}{4}$
	11	9	$7\frac{3}{4}$...7 far.
			10 [or $1\frac{3}{4}d.$
	£114	16	$7\frac{1}{4}$, *proof.*

farthings, or $1\frac{3}{4}d.$, remaining, which, with the coins of the realm, cannot be divided by 10, or, in other words, does not admit of a tenth part of it being taken. The entire result we have arrived at means that out of the given sum of money ten equal shares, each of which will be £11 - 9 - $7\frac{3}{4}$, can be taken; and that there will remain 7 farthings, which cannot be divided into ten equal shares. The proof is performed as in simple division.

Exercises.

	£	*s.*	*d.*		£	*s.*	*d.*		£	*s.*	*d.*
1.	10	11	$8\frac{1}{2} \div 2$	5.	8	13	$4 \div 6$	9.	5	12	$9 \div 11$
2.	2	16	$7 \div 3$	6.	1	19	$5\frac{1}{2} \div 7$	10.	20	1	$3 \div 12$
3.	4	5	$9\frac{3}{4} \div 4$	7.	3	15	$7 \div 8$	11.	1	0	$0 \div 6$
4.	9	9	$6\frac{1}{4} \div 5$	8.	5	17	$11\frac{1}{2} \div 9$	12.	1	0	$0 \div 7$

RULE II. *To divide by a number which is greater than 12, but is the product of two or more factors, each less than 13*: (1.) Divide the given quantity, according to the last rule, by one of the factors. (2.) Divide the quotient by another factor. (3.) Divide the result thus obtained by another factor, if there be so many: and thus proceed, whatever may be their number.

Exam 2. Divide £59 - 13 - $3\frac{1}{2}$ by 66.

In this example the factors are 6 and 11. In the division by 6 the quotient* is £9 - 18 - $10\frac{1}{2}$, and the remainder 2 farthings; and in the division of this quotient by 11, the quotient resulting is 18*s.* $0\frac{3}{4}d.$, and the remainder 9. This remainder being multiplied by 6, the first divisor, and the product increased by the former remainder 2 (*see page* 42), the true remainder is found to be 56 farthings, or 1*s.* 2*d.* In the proof by multiplication, this remainder must be added to the final product.

	£	*s.*	*d.*	
6)	59	13	$3\frac{1}{2}$	
11)	9	18	$10\frac{1}{2}$	...2
	£0	18	$0\frac{3}{4}$	...56 far.
			6	[or 1*s.* 2*d.*
	5	8	$4\frac{1}{2}$	
			11	
	59	12	$1\frac{1}{2}$	} add
		1	2	}
	£59	13	$3\frac{1}{2}$	, *proof.*

* In proceeding to work at questions in division of this kind, in which it is required to divide a given quantity into a number of equal parts, together, it may be, with a remainder smaller than any one of those equal parts, if the learner has for present purposes the misfortune to recollect the Latin derivation of the word *quotient*, from *quoties*, *how often*, according to which the word would properly apply to the answer to the question, *How often is one given number or quantity contained in another given number or quantity?* he ought now just to make an effort completely to ignore, in cases of the present kind, the derivation of the word. The English language does not at present supply us with any other word suitable for use instead of *quotient*, and so we must go on using that name,

In the use of this rule, if there be no pence in the dividend, or if it end in 3, 6, or 9 pence, and if one of the factors be 3, 6, or 12, it is better to use that factor first, as in the division by it there will be no remainder. If one factor be 2, 4, or 8, and there be no farthings in the dividend, it is generally better to begin with that factor. If 10*d.* be contained without remainder in the shillings and pence of the dividend (as in 1*s.* 8*d.*, 2*s.* 6*d.*, 3*s.* 4*d.*, 15*s.* 10*d.*, &c.), and one of the factors be 5 or 10, it is best to begin with that factor. The same may be observed in relation to 5*d.* and 2½*d.*

	Exercises.					*Answers.*			
	£	*s.*	*d.*			£	*s.*	*d.*	*Rem.*
13.	73	18	2	÷ 14	....	5	5	7	
14.	8	3	6	÷ 18	....	0	9	1	
15.	40	10	0	÷ 49	....	0	16	6¼	5¾*d.*
16.	36	11	8	÷ 16	....	2	5	8¾	
17.	58	7	0	÷ 36	....	1	12	5	
18.	59	11	11¼	÷ 27	....	2	4	1¾	
19.	48	8	2½	÷ 50	....	0	19	4¼	6*d.*
20.	9	14	8	÷ 32	....	0	6	1	
21.	31	2	8½	÷ 48	....	0	12	11½	8½*d.*
22.	50	0	0	÷ 63	....	0	15	10¼	1*s.* 2¼*d.*
23.	55	8	0¾	÷ 77	....	0	14	4½	1*s.* 2¼*d.*
24.	53	12	7½	÷ 81	....	0	13	2¾	1*s.* 0¾*d.*
25.	113	14	9	÷ 96	....	1	3	8¼	9*d.*
26.	53	10	6	÷ 120	....	0	8	11	6*d.*
27.	31	10	4	÷ 70	....	0	9	0	4*d.*
28.	38	10	0	÷ 88	....	0	8	9	
29.	124	16	6	÷ 84	....	1	9	8½	1*s.*
30.	100	0	0	÷ 121	....	0	16	6¼	11¾*d.*
31.	375	10	6	÷ 144	....	2	12	1¾	1*s.* 6*d.*
32.	138	15	0	÷ 150	....	0	18	6	
33.	111	2	6	÷ 180	....	0	12	4	2*s.* 6*d.*
34.	255	18	9	÷ 225	....	1	2	9	
35.	746	0	0	÷ 275	....	2	14	3	1*s.* 3*d.*

RULE III. *To divide by a number which is greater than* 12, *and is not produced by factors below* 13: The process is to be conducted as in Rule I., except that long division is to be employed instead of short.

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although in doing so we ought clearly to understand that any consideration of the proper etymological sense of the word would be completely misleading. For instance, if we divide £59 - 13 - 3 by 6, and say that the *quotient* is £9 - 18 - 10½, we are not to entertain for a moment the idea that 6 is contained in £59 - 13 - 3, £9 - 18 - 10½ times.

Exam. 3. Divide £2074 - 6 - 9¼ by 597.

Here, we commence by dividing £2074 by 597, and so we get £3 to be set down as the first part of the required quotient, and we have remaining £283. This number of pounds we reduce to a number of shillings by multiplying it by 20, and we add in the 6s., and so we get 5666s.; from which, as in simple division, we obtain for quotient 9s. The remainder 293s. is then reduced to pence, and increased by the 9d.: and, by continuing the operation in a similar manner, we have finally a remainder of 370 farthings, or 7s. 8½d. The work is proved by multiplying the quotient by 600 (6 × 10 × 10), and subtracting three times the quotient, to obtain the product by 597; and finally, by adding the remainder 7s. 8½d. to the result.

£	s.	d.	£	s.	d.
597)2074	6	9¼	(3	9	5¾, *answ.*
1791					6
283			20	16	10½
20					10
5666			208	8	9
5373					10
293			2084	7	6
12			10	8	5¼
3525			2073	19	0¾
2985				7	8½
540			£2074	6	9¼, *proof.*
4					
2161					
1791					
370 farthings, or 7s. 8½d.					

	Exercises.				*Answers.*			
	£	s.	d.		£	s.	d.	*Rem.*
36.	61	0	0	÷ 13	4	13	10	2d.
37.	9	3	7	÷ 17	0	10	9½	1½d.
38.	50	4	2	÷ 19	2	12	10	4d.
39.	99	0	0	÷ 26	3	16	1¾	2½d.
40.	115	12	6	÷ 37	3	2	6	
41.	93	9	4½	÷ 43	2	3	5½	8d.
42.	82	6	11	÷ 51	1	12	3½	½d.
43.	54	13	2	÷ 65	0	16	9¾	4½d.
44.	108	18	1½	÷ 71	1	10	8	9½d.
45.	167	16	0	÷ 85	1	19	5¾	3¼d.
46.	53	12	0	÷ 91	0	11	9¼	10¼d.
47.	344	14	0	÷ 103	3	6	11	1s. 7d.
48.	179	6	0	÷ 133	1	6	11½	6½d.
49.	599	2	4	÷ 201	2	19	7¼	1s. 10¾d.
50.	342	11	4½	÷ 313	1	1	10½	4s. 6d.
51.	400	0	0	÷ 365	1	1	11	5d.
52.	9846	13	4	÷ 5275	1	17	4	
53.	2045	16	5¼	÷ 4083	0	10	0¼	1s. 4½d.
54.	3982	2	8½	÷ 1347	2	19	1½	1s. 4d.
55.	5139	16	8	÷ 1819	2	16	6	£1 - 3 - 2

56. If a chest of tea, containing 96 lbs., cost £16, what will 1 lb. cost? *Answ.* 3*s.* 4*d.*

57. If a hundred-weight of sugar cost £2 - 2 - 0, what are the costs of a pound and of a stone? *Answ.* $4\frac{1}{2}$*d.* and 5*s.* 3*d.*

58. If a contribution of £354 - 11 - 6 is to be made up in equal shares by 26 persons, how much must each contribute? *Answ.* £13 - 12 - 9.

59. If prize money to the amount of £495 - 5 is to be divided equally among 75 seamen, how much will each receive? *Answ.* £6 - 12 - $0\frac{3}{4}$; *rem.* $3\frac{3}{4}$*d.*

60. If a person spend £200 a year, how much does he spend each day at an average? *Answ.* 10*s.* $11\frac{1}{2}$*d.*; *rem.* $2\frac{1}{2}$*d.*

61. Three persons purchase a ship for £12,000; the first taking one share, the second three shares, and the third five. How much do they severally pay? *Answ.* £1333 - 6 - 8, £4000, and £6666 - 13 - 4.

RULE. IV. *To divide one quantity by another of the same kind, the quantities being one or both compoundly expressed; or else each being expressed in one denomination, but the one being in a different denomination from the other.* Make any convenient reduction of the divisor, or of the dividend, or of both, that will bring both to one same denomination; and then proceed by simple division.

Exam. 4. If a person has £52 - 10 - 8 available for purchasing sheep, how many can he buy out of a flock offered at £1 - 18 - 6 each, and how much of the money will remain?

Here the question may be regarded as being: How often can the price of one sheep be taken out of the whole sum available, and how much money will remain over, being less than the price of one sheep? The quotient,* or *number of times*, so found will obviously be also the *number of sheep* that can be bought. To carry out the work, we first reduce the divisor and dividend each to pence; and then, on dividing 12608 pence by 462 pence, we get the quotient 27, which, as already shown, will be the number of sheep that can be bought; and we find that after the purchase of these, there will be 134 pence, or 11*s.* 2*d.*, remaining unexpended.

£ *s.* *d.*	£ *s.* *d.*
1 18 6	52 10 8
20	20
38	1050
12	12
462 pence	462)12608(27
	924
	3368
	3234
	12)134*d. rem.*
	11*s.*2*d. rem.*

* The name *quotient* is to be noticed as being properly applicable in accordance with its Latin derivation to the result in cases of this kind, as it is here a statement of *how often* one quantity is contained in another quantity.

Exer. 62. A merchant buying goods for exportation, has £97 - 6 - 8 available for purchasing watches: how many can he buy at £3 - 16 - 0 each, and how much of the money will remain over? *Answ.* 25 watches, and £2 - 6 - 8 money remaining.

63. How many wires, each 1 foot 5 inches long, for making bird-cages, can be cut out of a coil of wire 524 feet 9 inches long, and how much of the wire will remain, being too short for the purpose? *Answ.* 370 wires, and 7 inches of wire over.

64. How many links for a chain can be made out of a coil of wire 115 feet long, if the length wanted for each link is $\frac{3}{4}$ of an inch? *Answ.* 1840 links.

65. If a railway train travels 1 mile in 1 minute 32 seconds, how far will it go in 28 minutes at the same rate? *Answ.* $18\frac{6}{23}$ miles.

	Exercises.		*Answers.* *Quot.*	*Rem.*
66.	£256 - 16 - 9	÷ £9 - 9 - 11	27	9*s.*
67.	£62 - 13 - 2	÷ 13*s.* 4*d*	93	13*s.* 2*d.*
68.	520 yards	÷ 2 feet 7 inches	603	2 ft. 3 in.
69.	2 tons 14 cwt. 3 q.	÷ 2 cwt. 1 q. 17 lbs.	22	1 c. 3 q. 18 lbs.
70.	£3 - 11 - 6	÷ $5\frac{1}{2}$*d.*	156	

At the close of the introductory chapter on fractions, pages 54 and 55, some examples have already been given of operations called multiplying and dividing by "numbers containing fractions" (or rather by fractional numerics); and it may be useful here, in connection with the preceding chapters on compound multiplication and division, to give some examples and exercises of a similar kind for cases in which the multiplicand or the dividend is a compoundly expressed quantity.

Exam. 5. Required the price of $22\frac{3}{4}$ cwt. of pork, at £2 - 16 - 6 per cwt.?

Here we see that we shall get the price of $22\frac{3}{4}$ cwt. if we take $22\frac{3}{4}$ of the price of 1 cwt.; and the doing of this is called multiplying the price of 1 cwt. by $22\frac{3}{4}$. To perform the operation, we first take 22 of £2 - 16 - 6, or, in other words, we multiply that money by 22, doing so in the way already explained. Then we take $\frac{3}{4}$ of £2 - 16 - 6 (an operation which is called multiplying by $\frac{3}{4}$); and to do this we multiply, in a separate place, that money by 3, and divide the product by 4: the result is £2 - 2 - $4\frac{1}{2}$; which being added to the product by 22, the sum is £64 - 5 - $4\frac{1}{2}$, the product required. The work for the fractional part is left for the learner to perform. The answer might also have been found by multiplying by 23, and taking one fourth of £2 - 16 - 6 from the product. If the multiplier had been $22\frac{1}{4}$ or $22\frac{1}{2}$, we should have found the product by 22, as before; and in the first case have added to it one fourth, in the second, half of the multiplicand.

£	*s.*	*d.*	
2	16	6	
		2	
5	13	0	
		11	
62	3	0	
2	2	$4\frac{1}{2}$	
£64	5	$4\frac{1}{2}$,	*answ.*

Exam. 6. If $22\frac{3}{4}$ gallons of brandy be bought for £30 - 2 - $10\frac{1}{2}$, what is the cost of each gallon?

The work here will constitute what is commonly called *dividing by a fractional number*, which in this case is $22\frac{3}{4}$. The nature of such operations is often found rather difficult to understand completely. It may be clearly explained in the following way:—We see that the cost of the $22\frac{3}{4}$ gallons must be $22\frac{3}{4}$ of the cost of 1 gallon. Hence we want to find a quantity of money such that $22\frac{3}{4}$ of it shall be £30 - 2 - $10\frac{1}{2}$.* To carry out the work practically, we observe that if $22\frac{3}{4}$ gallons cost £30 - 2 - $10\frac{1}{2}$, 4 times $22\frac{3}{4}$ gallons would cost 4 times £30 - 2 - $10\frac{1}{2}$. Hence we multiply $22\frac{3}{4}$ gallons by 4, and find 91 gallons; and we multiply £30 - 2 - $10\frac{1}{2}$ by 4, and find £120 - 11 - 6. Then we see that 91 gallons would cost £120 - 11 - 6, or that 1 gallon would cost $\frac{1}{91}$ of £120 - 11 - 6; and so we divide £120 - 11 - 6 by 91, and find £1 - 6 - 6 for the required result.

$$
\begin{array}{rrrrrrr}
 & \text{£} & s. & d. & & & \\
22\frac{3}{4})\!\! & 30 & 2 & 10\frac{1}{2} & & & \\
4 & & & 4 & \text{£} & s. & d. \\
\hline
91)\!\! & 120 & 11 & 6 & (\,1 & 6 & 6 \\
 & 91 & & & & & \\
\hline
 & 29 & & & & & \\
 & 20 & & & & & \\
\hline
 & 591 & & & & & \\
 & 546 & & & & & \\
\hline
 & 45 & & & & & \\
 & 12 & & & & & \\
\hline
 & 546 & & & & & \\
 & 546 & & & & & \\
\hline
 & \cdots & & & & &
\end{array}
$$

	Exercises.					*Answers.*		
	£	s.	d.			£	s.	d.
71.	4	11	$6\frac{3}{4}$	$\times\ 17\frac{1}{3}$	....	79	7	1
72.	7	3	$2\frac{1}{4}$	$\div\ 21\frac{3}{4}$	....	0	6	7
73.	1	18	10	$\times\ 54\frac{5}{8}$	....	106	1	$3\frac{1}{4}$
74.	7	5	$11\frac{3}{4}$	$\div\ 11\frac{3}{8}$	....	0	12	10

* To show the connexion between this fractional division and ordinary division by a "whole number," we may observe that if the given number of gallons had been 22, we would have found the price of one gallon by taking $\frac{1}{22}$nd of the price of 22 gallons; that is, we would have had to find a quantity of money such that 22 of it would be £30 - 2 - $10\frac{1}{2}$; and this way of stating the operation for the whole number 22 is the same as the way in which the operation for the fractional number $22\frac{3}{4}$ has been stated here in the text above, and so the similarity of the two processes, both called division, is obvious.

We may further show, in the following slightly different way, why the operation in the text may be called division. The operation may obviously be interpreted as a case in which we virtually *divide* the given sum of money, which is the cost of the whole $22\frac{3}{4}$ gallons, into the 22 equal portions which would pay for the 22 gallons each by itself, and the one portion besides which would pay for the additional $\frac{3}{4}$ gallon.

RATIO, RATE, AND PROPORTION.

INTRODUCTORY REMARKS.—(1.) In order that the pupil may be sufficiently prepared to understand the teaching of RATIO, RATE, and PROPORTION here, he ought to be previously very well acquainted with the explanations of fractions already given in the Introductory Chapter on Fractions, and in earlier passages referred to in that chapter. He may also with advantage, if convenient, carefully read beforehand, at this stage, some portions of the advanced chapter on fractions given farther on in this treatise; especially those which relate to the processes commonly called multiplying and dividing by fractions, or by fractional numerical expressions; but it is not essentially necessary to read those passages beforehand. (2.) When in questions or operations in arithmetic an unknown quantity, or number, or numeric generally, has to be spoken of or dealt with, the verbal statements and the operations may often be greatly facilitated by denoting that unknown by a letter. Often, also, explanations and demonstrations may be greatly facilitated and abbreviated by using letters as representatives of quantities or numerics, whether known or unknown. This method of making things easy, which is regularly used in Algebra, is deserving of being introduced early to pupils of Arithmetic in schools; and it will be used occasionally, when convenient, in the present treatise. It will be used in several of the explanations in the present chapter.

DEFINITION OF RATIO.—In comparing two quantities of any one kind of thing, or two numbers, or two numerics generally, the numeric * that either of them is of the other is called the RATIO of that one to that other.

Thus, in comparing 16 inches with 2 feet (that is, comparing one quantity of length with another), we may readily see that the first is $\frac{2}{3}$ of the second; then the numeric $\frac{2}{3}$ is the ratio of the first to the second: and we may readily see that the second is $\frac{3}{2}$ of the first; and so the numeric $\frac{3}{2}$ is the ratio of the second to the first. Again, in comparing the length $2\frac{1}{2}$ feet with the length 6 inches, we

* The meaning of the word *numeric* will be found explained in the chapter of Introductory Explanations, at page 4, foot note; and further in the Introductory Chapter on Fractions, at page 49. Also an example of one quantity expressed as a numeric of another will be found treated in Example 8, pages 52 and 53.

observe that the first quantity is 5 of the second; then the numeric 5 is the *ratio* of the first to the second.*

Also in comparing the number 22 with the number 11, we see that the first is 2 of the second; and so 2 is the ratio of 22 to 11. In comparing 5 with 7, we see that 5 is $\frac{5}{7}$ of 7; and so $\frac{5}{7}$ is the ratio of 5 to 7. In comparing $\frac{2}{3}$ with $\frac{7}{9}$, we may easily find that the first is $\frac{6}{7}$ of the second, and so $\frac{6}{7}$ is the ratio of the first to the second. To show the truth of the statement that the first in this case is $\frac{6}{7}$ths of the second, we may observe that instead of $\frac{2}{3}$ we may take its equal $\frac{6}{9}$, and then, having got the same denominator, 9, in both the fractions to be compared, we can see obviously that 6 ninths is $\frac{6}{7}$ of 7 ninths (just as 6 inches is $\frac{6}{7}$ of 7 inches). This example, showing the ratio of a fraction to a fraction, is introduced here in order to complete the illustration of the meaning of the definition which has been given of ratio. Detailed explanations in respect to ratios of fractional numerics, and as to practical modes of treating them, will, however, be for the most part deferred till after the advanced chapter on fractions, because a pupil having arrived at that stage will be better prepared than here to follow out such explanations.

When it is two proper numbers (integers) that we wish to compare, we can always find what numeric the first is of the second by taking a fractional expression less or greater than unity, as the case may be, having the second for its denominator and the first for its numerator, and that fractional expression will be the required numeric; and so will be the ratio of the first to the second. In some cases the fractional expression so found as the ratio, will be reducible to a proper number: thus the ratio of 12 to 3 we see is the fraction $\frac{12}{3}$, which is reducible to the number 4; and so we say the ratio of 12 to 3 is 4.

The two quantities, or numbers, or numerics generally, which are compared as to their ratio, are called the TERMS of the ratio. The first is called the ANTECEDENT, and the second the CONSEQUENT.

The name PROPORTION has many varieties of signification in general language. It is ordinarily applied in arithmetic, and other branches of mathematics, to re-

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* In this case we might say the *number* 5 is the ratio of the first to the second, but the word *numeric* is here preferred, because it allows of our having one brief and easily intelligible definition and method of explanation, that will suit alike for all ratios in arithmetic, whether they be what are commonly called whole numbers; or whether they be fractions, proper or improper; or whether, when greater than unity, they be expressed by what are commonly called mixed numbers. For want of any single word by which to name alike all such numerical expressions as have just now been referred to, it has been customary to use in various cases, in defining or explaining what is meant by a ratio, some one or more of the following names, or occasionally others besides:—number, whole number, fraction, fractional number, proper fraction, improper fraction, mixed number, quotient, multiple, part or parts, multiple-part. If the advanced reader will compare the definition and explanations of *ratio* given here with those given in other books, he may perceive great advantages both in brevity and in clearness attained through the use of the word *numeric*.

lations among quantities or numerics, or among quantities and numerics, which involve the consideration of *equality of ratios* or the treatment of *equal ratios.*

DEFINITION OF RATIO-EQUATION.—A statement of equality of two ratios is very commonly called a PROPORTION or an ANALOGY; but these names are often used with some haziness in the ideas associated with them, and they are even sometimes not restricted to the exact signification of a statement of equality of two ratios. Hence, and for other reasons, a statement of equality of two ratios may, with advantage in clearness and precision, be called by the self-explanatory name, a RATIO-EQUATION.

For example, the ratio of 10 to 15, and the ratio of 8 to 12 are equal (each being $\frac{2}{3}$, because 10 is $\frac{2}{3}$ of 15, and 8 is $\frac{2}{3}$ of 12); and a statement of this equality constitutes a *ratio-equation.* A ratio-equation is sometimes written thus:—

$$10 : 15 = 8 : 12,$$

which may be read, *Ratio of* 10 *to* 15, *equal to ratio of* 8 *to* 12; but it is more usually written thus:—

$$10 : 15 :: 8 : 12;$$
$$\text{or, As } 10 : 15 :: 8 : 12.$$

In the former of these last two ways it is often read, 10 *is to* 15 *as* 8 *is to* 12; and in the latter of the two it is read, *As* 10 *is to* 15, *so is* 8 *to* 12, the notion of equality of ratios being not expressly adduced in these last two ways of speaking. In these various modes two dots are put between the terms of each ratio to indicate their connection, and either the sign of equality is interposed between the equal ratios to denote their equality, or else four dots are interposed to mark exactly the same meaning.

ANOTHER DEFINITION OF RATIO-EQUATION.—The definition just given of a PROPORTION, or an ANALOGY, or a RATIO-EQUATION, is to the effect that, for a ratio-equation, *what numeric the antecedent in the first ratio is of its consequent, the same numeric must the antecedent in the second ratio be of its consequent.* We may as well, and quite to the same effect, and sometimes more conveniently, say that for a ratio-equation, *what numeric the consequent in the first ratio is of its antecedent, the same numeric must the consepuent in the second ratio be of its antecedent.*

That these two definitions are really equivalent may be understood from consideration of examples such as the following. For

brevity we may call the two terms of the first ratio A and B, where these letters signify either two numerics or two quantities like in kind one to the other; and we may call the two terms of the second ratio C and D, where likewise these letters signify either two numerics or two quantities like in kind one to the other, though they may be unlike in kind to the previous pair A and B. Then, for instance, let us suppose we know that B is $\frac{7}{5}$ of A, and that D is $\frac{7}{5}$ of C; so that in one pair the latter term is the same numeric of the former, as the latter is of the former in the other pair. Now we may note in form of an equation

$$B=\tfrac{7}{5}A$$

and by multiplying both sides of this equation by the denominator 5, we get a new equation—

$$5B=7A;$$

and next, by dividing both sides of this by 7, we get from it another new equation—

$$\tfrac{5}{7}B=A.$$

And this is to say that A is $\frac{5}{7}$ of B.

By going through exactly the same process in respect to C and D, commencing with the given equation

$$D=\tfrac{7}{5}C,$$

we would find that

$$C=\tfrac{5}{7}D.$$

Thus we see that in respect to the two pairs of terms, A and B, with C and D, since it was given that what numeric the latter term is of the former in the one pair ($\frac{7}{5}$ in our example) the same numeric is the latter term of the former in the other pair ($\frac{7}{5}$ in our example), it follows that what numeric the former term is of the latter in the one pair ($\frac{5}{7}$ in our example), the same numeric ($\frac{5}{7}$ in our example) is the former of the latter in the other pair; or, in other words, the ratio of the former to the latter of the one pair is equal to the ratio of the former to the latter of the other pair. So the second definition given of ratio-equation is reducible to the first, or they are interchangeable, and either may be used in preference to the other, according to temporary convenience.

What has just been proved may be briefly stated, for future reference, in the form of a proposition, as follows:—

Proposition 1. When two ratios are such that the consequent in one is the same numeric of its antecedent as the consequent in the other is of its antecedent, then also they are such that the antecedent in either of them is the same numeric of its consequent as the antecedent in the other is of its consequent, and the two ratios are equal.

Proposition 2. The ratio of one quantity to another is equal to the ratio of the numeric expressing the one

quantity in any denomination to the numeric expressing the other quantity in the same denomination.

Thus the ratio of 5 yards to 3 yards is equal to the ratio of 5 to 3. It is $\frac{5}{3}$, because 5 is $\frac{5}{3}$ of 3.

And, as a further example, it may be seen that the ratio of $2\frac{1}{4}$ yards to $6\frac{3}{4}$ yards is equal to the ratio of $2\frac{1}{4}$ to $6\frac{3}{4}$, and that it is $\frac{1}{3}$, because $2\frac{1}{4}$ is obviously $\frac{1}{3}$rd of $6\frac{3}{4}$.

And, as yet another example, we may propose to find some simple expression for the ratio of $5\frac{1}{2}$ yards to 3 feet 4 inches. To do this we may obviously, according to the proposition (Prop. 2), commence by reducing the given expressions for the two quantities of length to new expressions in one same denomination. We may in this case conveniently select the denomination of inches as being the lowest occurring in either of the two given expressions. Then we can find easily that the two lengths $5\frac{1}{2}$ yards and 3 feet 4 inches may be expressed as being 198 inches and 40 inches; and so the ratio of the former length to the latter length may be stated as being the ratio of the number 198 to the number 40. Or the required ratio is $\frac{198}{40}$ or $\frac{99}{20}$.

Proposition 3. From the foregoing it follows obviously that—In a ratio-equation stating two equal ratios between two pairs of quantities, if the two terms of the first ratio be expressed in one denomination, the same for both; and if the two terms of the second ratio be expressed in one denomination, the same for both; then the numerics so expressing the ratio-equation will themselves, when kept in the same order, constitute a ratio-equation.

Thus in the ratio-equation

Ratio 4 shillings to 5 shillings = ratio 12 yards to 15 yards;
or, 4 shillings : 5 shillings :: 12 yards : 15 yards,

we have also the numerical ratio-equation

$$4 : 5 :: 12 : 15.$$

The significance and importance of the proposition here set forth will be more fully appreciated through considerations such as the following. We may notice that the same quantitative ratio-equation as before may be expressed thus:—

Ratio 4 shillings to 60 pence = ratio 432 inches to 45 feet;
or, 4 shillings : 60 pence :: 432 inches : 45 feet.

But the numerics or the numbers,

4, 60, 432, and 45,

here occurring have not an equality between the ratio of the former to the latter in the first pair, and the former to the latter in the

second pair; there is not an equality between the ratios of 4 to 60, and 432 to 45; or it is not the case that

$$4 : 60 :: 432 : 45.$$

DEFINITIONS.—Four numbers (or four numerics generally) constituting two equal ratios, and written in order so that the ratio of the first to the second is equal to the ratio of the third to the fourth (as, for instance 10, 15, 8, 12), are commonly called *four proportionals.* The first and last of the four terms are called the *extremes*, and the middle two are called the *means.*

In a numerical ratio-equation the fourth term is often spoken of as the *fourth proportional to the other three terms.* Thus, for instance, when two numbers are given as the antecedent and consequent of one ratio, and a third number is given as the antecedent for an equal ratio, and it is required to find the consequent for that ratio, the requirement is often, for brevity, spoken of as being *to find a fourth proportional to three given numbers.*

RULE I. The fourth proportional to three given numbers* may be found by multiplying together the second and third, and dividing the product by the first. The result, including the fraction due to the remainder, if any, in the division, will be the required fourth proportional.

To illustrate this, let it be required to find the fourth proportional to 3, 2, and 12. Here if we put the letter x to denote the fourth proportional, as yet unknown, we may write the ratio-equation—

$$\text{Ratio } 3 : 2 = \text{ratio } 12 : x;$$
$$\text{or, } 3 : 2 :: 12 : x.$$

Now, according to the second definition of ratio-equation, given on page 95, we see that what numeric 2 is of 3, the same numeric

* The word *numbers*, as here used, is to be understood as meaning *numbers properly so called*; that is, *integers.* The rule, however, would still hold good if, instead of *numbers*, we were to insert the word *numerics*, and so were to extend the scope of the rule to the finding of the fourth proportional to any three numerics, whether whole or fractional. But, as the pupil at the present stage may be supposed not to be well acquainted with the management of fractional numerics, except in very simple operations, the scope of the rule is *for the present* limited to the case in which the three given terms for the ratio-equation are all integers. The required fourth term, being found by a process of division, will, of course, usually come out to be fractional. The rule as extended to the case in which a fourth proportional to three numerics, whether whole or fractional, is to be found, will be explained farther on in this treatise.

must x be of 12; but 2 is $\frac{2}{3}$ of 3; and so x must be $\frac{2}{3}$ of 12, or we have

$$x=\frac{2\times 12}{3},$$

$$\text{or, } x=8;$$

and so we see that the required fourth proportional to the three given numbers is obtained by multiplying together the second and third and dividing the product by the first.

To illustrate the rule further, let it be required to find the fourth proportional to 4, 5, and 13. Here we may write the ratio-equation—

$$4:5::13:x.$$

Now we see that what numeric 5 is of 4, the same numeric must x be of 13; but 5 is $\frac{5}{4}$ of 4, and so x must be $\frac{5}{4}$ of 13; or we have

$$x=\frac{5\times 13}{4};$$

and so again we see that to find the required fourth proportional we are to multiply the second and third terms together, and to divide their product by the first. The work may conveniently be carried out in the arrangement shown in the margin here, where the answer is found to be $16\frac{1}{4}$.

$$\begin{array}{r} 4:5::13:x \\ 5 \\ \hline 4)\overline{65} \\ x=16\tfrac{1}{4}, \textit{ answ.} \end{array}$$

Exam. 1. Find the fourth proportional to 21, 12, and 14. Here we may put x to denote the required fourth proportional, and we may proceed as in the margin, commencing with the statement that: As 21 is to 12, so is 14 to x. Then, by multiplying the second and third together, and dividing the product by the first, we get 8 as the value of x; or we find that the fourth proportional required is 8.

$$\begin{array}{r} 21:12::14:x \\ 12 \\ \hline 21)\overline{168}(8=x, \\ 168 \textit{ answ.} \\ \hline \ldots \end{array}$$

Exam. 2. Find the consequent for a ratio of which the antecedent is 38, and which shall be equal to the ratio of 7 to 23.

Here putting x to denote the required consequent, we may write, as in the margin, the ratio-equation—

$$7:23::38:x.$$

Then multiplying the second and third terms together, and dividing their product by the first, we get for the required term $124\frac{6}{7}$. It is good to notice that since the consequent in the first ratio is $\frac{23}{7}$ of its antecedent, the consequent in the second ratio must be $\frac{23}{7}$ of its antecedent: and so we find x by taking $\frac{23}{7}$ of 38; or we take $\frac{1}{7}$th of 23 times 38. By this view the reasons of the process in the margin may be clearly seen.

$$\begin{array}{r} 7:23::38:x \\ 23 \\ \hline 114 \\ 76 \\ \hline 7)\overline{874} \\ \hline 124\tfrac{6}{7}=x, \\ \textit{answ.} \end{array}$$

Exercises.

1. Find the fourth proportional to the following three numbers:—2, 5, 16. *Answ.* 40.

2. Find the value of x, so that

$$18 : 21 :: 24 : x. \quad \textit{Answ.}\ x = 28.$$

3. Given 35 and 402 as the first and second terms of a ratio; and given 365 as the first term for an equal ratio; find the second term for that second ratio. *Answ.* $4192\frac{2}{7}$.

4. What is the numeric that 13 is of 15? *Answ.* $\frac{13}{15}$.

5. What is the numeric that 18 is of 3? *Answ.* 6; because 18 is 6 of 3.

6. Find a numeric x, such that what numeric 18 is of 3, the same numeric shall x be of 5. *Answ.* 30; because 18 is 6 of 3, and so x must be 6 of 5, or x must be 30.

7. Find a numeric x, such that what numeric 18 is of 5, the same numeric shall x be of 7. *Answ.* $25\frac{1}{5}$; because 18 is $\frac{18}{5}$ of 5, and so x is to be $\frac{18}{5}$ of 7, or $x = \frac{18 \times 7}{5}$.

8. Find x such that

$$\text{Ratio } 14 : 19 = \text{ratio } 8 : x. \quad \textit{Answ.}\ x = 10\tfrac{6}{7}.$$

Because, using the second definition of ratio-equation (page 95), we may say 19 is $\frac{19}{14}$ of 14, and so x must be $\frac{19}{14}$ of 8, or $x = \frac{19 \times 8}{14}$. Or we might at once write, $14 : 19 :: 8 : x$; and then by Rule I. multiply together the second and third terms, and divide the product by the first.

9. Find x so that $243 : 22 :: 46 : x$. *Answ.* $x = 4\frac{40}{243}$.

10. Find x so that $100 : 5 :: 22 : x$. *Answ.* $x = 1\frac{1}{10}$.

11. Find x so that $365 : 35 :: 146 : x$. *Answ.* $x = 14$.

12. Find x so that $112 : 48 :: 56 : x$. *Answ.* $x = 24$.

13. Find the consequent for a ratio of which the antecedent is 521, and which is to be equal to the ratio of 46 to 29.

Answ. $328\frac{21}{46}$.

14. Find some simple numerical expression for the ratio of £1 to 6*s.* 8*d.* *Answ.* 3; because one pound is 3 of 6*s.* 8*d.*

15. Find some simple numerical expression for the ratio of 54 minutes 18 seconds to 1 hour 7 minutes 10 seconds. *Answ.* $\frac{3258}{4030}$, or $\frac{1629}{2015}$.

16. Find some simple numerical expression for the ratio of 2*s.* 4*d.* to 21*s.* *Answ.* $\frac{28}{252}$, or $\frac{14}{126}$, or $\frac{7}{63}$, or $\frac{1}{9}$.

Brief Explanations on Rate and Rate-Equivalence.—In the statement of questions which are suitable for having their answers worked out by some of the most usual applications of the method of proportion, or we may say, by use of a ratio-equation in which one term is required to be found, the expression "*at the same rate*" is often introduced, or is often implied without being formally

stated. The exact meanings of *rate* and *sameness or equivalence of rate* will be explained further on. They cannot be completely explained very briefly; but, to allow of an early commencement being made by the pupil in the working out of some questions practically, it is convenient here briefly to indicate the meaning of the phrase "*at the same rate*" by a few examples.

In case of its being said that when 2 pounds of sugar is sold for 5 pence, the sugar is sold *at the same rate* as when 6 pounds is sold for 15 pence, the statement signifies that the ratio of the one quantity of sugar to the other is equal to the ratio of the one quantity of money to the other in corresponding order. Again, when it is said that if 15 men reap an acre in a day, 5 men would reap $\frac{1}{3}$ of an acre in a day, *at the same rate*, it is implied that the one number of men is to the other number of men as the area reaped in a day by the first set of men is to the area reaped in a day by the second set of men; or, in other words, that the ratio of the first number of men to the second number of men is equal to the ratio of the area reaped in a day by the first set of men to the area reaped in a day by the second set of men.

Temporary Rules for Use at this Stage.

(1.) When there are given a quantity of any commodity and its cost, and another quantity of the same for which we have to find the cost at the same rate, we may say: As the first quantity is to the second quantity, so is the cost of the first quantity to the cost of the second quantity.

(2.) When there are given a quantity of any commodity and its cost, and we are asked to find how much of the same thing would be bought for a stated cost at the same rate, we may say: As the one sum of money is to the other sum of money, so, in corresponding order, is the one quantity of goods to the other quantity of goods; and in arranging these terms it is convenient to place *first* the given cost of the given quantity of goods, and *second* the given cost of the required quantity of goods.

Examples relative to these two Temporary Rules.

Exam. 3. If 12 yards of ribbon cost 4 shillings, what would 15 yards cost at the same rate?

Here we may denote the required cost as x shillings, and we may conveniently place it fourth in order, and place the given cost third in order; and so we may say: As the quantity of ribbon for which the cost is given, is to the quantity for which the cost is required, so is the given cost to the required cost; or—

As 12 yards : 15 yards :: 4 shillings : x shillings.

Instead of this quantitative ratio-equation (stating equality between the ratio of one length to another, and the ratio of one cost to another) we may, according to Proposition 3, use the mere numerical ratio-equation—

$$12 : 15 :: 4 : x.$$

Then, as in the margin, we multiply the second and third terms together, and so find their product to be 60. Then, dividing this product by the first term, we get 5 as the value of x, and so the required cost is 5 shillings.

$$\begin{array}{l} 12 : 15 :: 4 : x \\ \quad\;\; \underline{\;\;4} \\ 12)\overline{60} \\ \quad\;\; \overline{\;5} = x \end{array}$$

Exam. 4. If 7 ℔s. of butter cost 103 pence, how much butter would be bought for 240 pence at the same rate?

This example obviously comes under the Temporary Rule (2). So we set down the proportion—

As 103 pence : 240 pence :: 7 ℔s. : x ℔s.

Then using, as in the margin, merely the numerical proportion, as we may do according to Proposition 3, we multiply the second and third terms together and divide by the first; and thus we find $x = 16\frac{32}{103}$. Hence the answer is $16\frac{32}{103}$ ℔s.

$$\begin{array}{l} \text{pence} \;\; \text{pence} \;\; \text{℔s.} \;\; \text{℔s.} \\ 103 : 240 :: 7 : x \\ \qquad\quad \underline{\;\;7} \\ 103)1680(16\frac{32}{103} = x \\ \qquad \underline{103} \\ \qquad\; 650 \\ \qquad\; \underline{618} \\ \qquad\;\; 32 \end{array} \qquad 16\tfrac{32}{103}\text{ ℔s., } \textit{answ.}$$

Exam. 5. If 2 cwt. 1 q. 31 ℔s. of sugar cost £3 - 16 - 8, what would 1 cwt. 3 q. 26 ℔s. cost at the same rate? This question obviously, from its data and requirement, falls under the Temporary Rule (1); and so, putting X to denote the required sum of money, we may say:—

As 2 cwt. 1 q. 27 ℔s. : 1 cwt. 3 q. 26 ℔s. :: £3 - 16 - 8 : X.

This is a quantitative, not a numerical ratio-equation; and, as the three given terms are compoundly expressed, we may, under guidance of Prop. 3, reduce the terms of the first ratio to one denomination the same for both, and we may reduce the given antecedent in the second ratio to one denomination. So we may reduce the terms of the first ratio to ℔s., as being the lowest denomination in either of them; and we may reduce the antecedent in the second ratio to pence, as being its lowest denomination. Thus—

2 cwt. 1 q. 27 ℔s. :	1 cwt. 3 q. 26 ℔s. ::	£3 - 16 - 8 : X
112	112	20
28	84	76
27	26	12
244	112	920 pence.
299 ℔s.	222 ℔s.	

And putting now x pence to denote the required sum of money, which was before denoted by the capital letter X as a quantity of

money irrespective of any particular denomination or denominations, and understanding that x denotes merely a numeric, we get our ratio-equation reduced to the form here set forth anew in the margin. Then using this ratio-equation now only in its numerical character, we multiply together the second and third terms, and divide the product by the first; and so we find $683\frac{23}{299}$ as the value of x. That is to say, the required sum of money is $683\frac{23}{299}$ pence. But, as the fraction $\frac{23}{299}$ of a penny is much less than a farthing, we may state the answer approximately as 683 pence, or £2 - 16 - 11.

```
299 lbs. : 222 lbs. :: 920 pence : x pence
                920
               ----
               4440
             1998
             ------
    299)204240(683 23/299 = x
        1794
        ----
         2484           12)683
         2392           2,0)5,6 - 11
         ----           ------------
          920             £2 - 16 - 11 23/299,
          897            or approximately
          ---             £2 - 16 - 11, answ.
           23
```

Exercises.

17. If 171 cwt. of sugar cost £216, what would 95 cwt. cost at the same rate? *Answ.* £120.

18. If 385 yards of linen cost £63, how much might be bought for £18 at the same rate? *Answ.* 110 yards.

19. How much wine may be bought for £396, if 90 gallons cost £72? *Answ.* 495 gallons.

20. If the yearly rent of a farm of 182 acres be £273, what is the rent of a part of it containing 42 acres? *Answ.* £63.

21. If 275 reams of paper cost £330, what would 990 reams cost? *Answ.* £1188.

22. If 378 yards of linen cost £54, how much might be bought for £23 - 10 - 6 at the same rate? *Answ.* Between 164 and 165 yards, or exactly $164\frac{27}{40}$ yards.

23. If 22 yards of ribbon cost 31 shillings, what will 15 yards cost at the same rate? *Answ.* $21\frac{3}{22}$ shillings, or approximately $21s.\ 1\frac{1}{2}d.$

24. If 148 gallons of rum cost £119 - 10, how much may be bought for £89 - 12 - 6? *Answ.* 111 gallons.

25. If £114 be paid for 52 cwt. 1 qr. 4 lbs. of flour, what would 122 cwt. cost at the same rate? *Answ.* £266.

26. What is the rent of 21 acres, 3 roods, 20 perches of ground, if the rent of 36 acres 3 roods be £42? *Answ.* £25.

PROPOSITION 4. If three "whole numbers" (integral numerics) are given for the first, second, and third terms of four proportionals; and if from these three the fourth proportional is found, which, as fixed by the data, may be integral or fractional; then, while these four proportionals, taken in their stated order, will con-

stitute a ratio-equation, another ratio-equation will be obtained if the second and third terms be interchanged. Thus if a, b, and c denote three given whole numbers, and if x denotes their fourth proportional, so that

$$a : b :: c : x,$$

then also $$a : c :: b : x.$$

The truth of this will be seen; because, if we proceed in either case to find the fourth proportional from the previous three terms, we have (according to the principle stated in Rule I.) to multiply together the second and third terms, and to divide their product by the first. But the interchange of the second and third terms will not alter their product, and therefore it will not alter the fourth proportional. That is to say, the second and third terms may be interchanged, and a new ratio-equation will so be obtained without change in the value of any of the four terms; and thus the proposition stated is proved to be true.

Remark.—The statement and proof have been here limited to the simple case in which the first three terms are "whole numbers." The more general proposition, it may be mentioned at present, is also true that from any numerical ratio-equation, in which any or all of its terms may be fractional, a new ratio-equation will result if the second and third terms be interchanged. This will be easily provable when the pupil is more familiar with the management of fractions than he is expected to be at this stage.

PROPOSITION 5. In a numerical ratio-equation having its first three terms "whole numbers," the product of the extremes is equal to the product of the means. The truth of this follows obviously from the principle stated in Rule I.; for, since the fourth term is obtainable by dividing the product of the means by the first term, that same number which was the product of the means would result again by multiplying the fourth term by the first, or, in other words, by taking the product of the extremes.

Remark.—In the case of this proposition again, correspondingly with the remark on the previous proposition, it may be mentioned that the more general proposition is also true—that in any numerical ratio-equation in which any or all of its terms may be fractional numerics, the product of the extremes is equal to the product of the means; but the simpler case alone, in which the first three terms are "whole numbers," is offered with proof to the pupil at the present stage.

PROPOSITION 6. A ratio is not altered by multiplying or dividing both its terms by the same number.

Thus if, for an example, we take the ratio of the first term to the second as being $\frac{5}{7}$, or, in other words, if the first term is $\frac{5}{7}$ of the second, then obviously twice the first term will be $\frac{5}{7}$ of twice the second term; three times the first term will be $\frac{5}{7}$ of three times the second term; and so on, when we multiply both terms by any number whatever. Also one half of the first term will obviously be $\frac{5}{7}$ of one half of the second term; one third of the first term will be $\frac{5}{7}$ of one third of the second term; and so on, when we divide both terms by any number whatever. Like illustrations might be given for the case of any other ratio of the first to the second term than the ratio here taken, which was $\frac{5}{7}$; and so the truth of the proposition may be regarded as established. Or it might, if preferred, be proved in general quite briefly by aid of algebraic notation.

Exercises.

27. In the ratio-equation $7 : 13 :: 9 : x$, find the value of x; and show that the fourth proportional will come out the same for a new ratio-equation having 7, 9, and 13 for its first, second, and third terms. *Answ.* The fourth proportional in each case is $\frac{13 \times 9}{7}$, or is $16\frac{5}{7}$.

28. In the ratio-equation $4 : 21 :: 19 : 99\frac{3}{4}$, find the product of the means and the product of the extremes. *Answ.* Product of means is 19×21, which is equal to 399; and product of extremes is $99\frac{3}{4} \times 4$, which is also equal to 399.

29. Find, according to the principle laid down in Prop. 6, one or more expressions for the ratio of 240 to 90, in smaller terms. *Answ.* Ratio of 24 to 9 or of 8 to 3.

30. Using the principle laid down in Prop. 6, express in "whole numbers" the ratio of $7\frac{3}{4}$ lbs. of butter to 12 lbs. of butter. *Answ.* Ratio of 31 to 48.

Variable Things, or Variables.—A thing is said to be *variable*, or it is called *a variable*, when it admits of being increased or diminished—that is, *varied* in quantity or in number. Quantity of cloth, for instance, is variable, because we may take either more or less of it. It may be 5 yards, or 20 yards, or $3\frac{1}{4}$ yards, or any other quantity, and the cost of the cloth will of course vary with the quantity of cloth. A *variable* may be either a quantity of something continuously variable—as, for instance, *length*, *time*, or *mass*—or else it may be a group of objects reckoned by number—as, for instance, a group of men, or of horses, or of tea-cups, none of which admit of being regarded as varying by fractions of a single one of the objects.

One Variable Proportional to another.—When two variables, whether of the same or of different kinds, are so connected that different particular values* of the one have each a particular value

* When *particular values* of a variable are spoken of in arithmetic and in mathematics generally, the word value is used in a special sense very different from its more ordinary sense of worth, in which we say that the value of a house is £2000, or the value of a horse is £90. A value of a variable usually signifies any one particular quantity or number which that variable may become, or which may be assigned to it. An example or two may best explain this. If water is flowing out of a cistern by a hole in the bottom, the quantity of

of the other corresponding with it; * and that whenever two values of the one are taken with two corresponding values of the other, the ratio of the second to the first value of the one variable is equal to the ratio of the second to the first value of the other variable; then either of these two variable things is said to be proportional to the other, or the two are said to be *mutually proportional* each to the other.

For example, if a railway train is moving at an unchanging velocity of 30 miles per hour, or of half a mile per minute, or of 44 feet per second, which all express the same velocity, the distance travelled from any fixed point on the line is proportional to the time elapsed from the moment of passing that point. The distance travelled and the time elapsed are two variables, each proportional to the other; or, in other words, they are mutually proportional.

Rate and Rate-Equivalence.—A few brief explanations in respect to the meaning of the phrase "*at the same rate*" have been already given (pages 100 and 101); and further explanations will now be offered. The subject of rate and rate-equivalence is of great importance. With a view to the development of clear ideas in respect to this subject, attention will be called to a few particular examples at first; and more general explanations will be afterwards adduced.

When we say that a certain kind of ribbon is sold at 9*d*. per yard, the expression *nine pence per yard* is a statement of a *rate*. This would be the same rate as *three shillings per four yards.*

We may say of a steamer that she is going at the rate of *2 miles per 10 minutes*; and we may ask how far she would go per hour at the same rate. Now, under the meaning of the expression *same rate*, it is here implied that she would go another length of 2 miles in another period of 10 minutes, a third length of 2 miles in a third period of 10 minutes; and so in 1 hour, which is six periods of 10 minutes, she would go six lengths of 2 miles each, or she would go 12 miles in an hour, or would be going *at the rate of 12 miles per hour.* Thus assuming the speed, or velocity, or what may be called the *time-rate* of motion, of the steamer to remain constant, or un-

---

water in the cistern is diminishing. The quantity in the cistern thus is a *variable*, not a *constant*; but the quantity in the cistern at any one moment is called a *particular value*, or simply a *value* of that variable. Again, the height of the tide water over an entrance sill of a dock increases and diminishes continuously as time advances. The height of the tide water over the sill is thus a variable, and the height at any moment is a value of that variable.

* The statement here in the text is carefully framed, so as not to say or to imply that *every* particular value of the one variable must have necessarily a particular value of the other variable corresponding with it. At a saw-mill, for instance, poles all alike except in length may be sold at so much per foot, up to some stated length, beyond which the price per foot may be higher, till another limit of length is reached beyond which none can be supplied. Again, if we were to say that the number of sovereigns that can be coined out of bullion is proportional to the weight of the bullion, we still could not say that for *every* particular weight of bullion there is a particular number of sovereigns corresponding to it; but we can say that whenever two particular quantities of the bullion are taken with two corresponding numbers of sovereigns, the ratio of the first to the second quantity of bullion is equal to the ratio of the first to the second number of sovereigns.

changing, we may designate that rate of motion as being 2 miles per 10 minutes, or 12 miles per hour; and obviously we might otherwise designate the same rate of motion as being 1 mile per 5 minutes, or 1056 feet per minute, or $17\frac{3}{5}$ feet per second. So much distance per so much time may be spoken of as a time-rate of motion. There may be other rates of motion besides time-rates. There may, for instance, in the case of a steamer, be the distance travelled per ton of coal consumed.

In each of the examples hitherto adduced the rate is between two *different* kinds of things: in one it is between a quantity of ribbon and a quantity of money; in another it is between a distance traversed and the time taken in traversing that distance; and in the remaining one it is between a distance traversed and the coal consumed for that distance. In the example which will next be given the rate will be between two quantities of the same kind of thing differently considered or differently applied. Thus:—

When income tax for a year is at 3 pence tax per £1 of income, we may say that the tax is at the rate of 3*d.* per £1; and the rate here is between two quantities of the same kind of thing, namely, money; but they are differently considered and differently dealt with, and it is this difference of character that leads to the relation between the two being usually spoken of as a *rate*. In this and other cases, however, in which for any reason there is so much of one kind of thing per so much of the same kind of thing, we may treat the relation between the two, according to convenience, either as a *rate* or as a *ratio*. Thus we can quite well say that the ratio of the income to the tax is the ratio of £1 to 3*d.*, or of 240*d.* to 3*d.* But 240 pence is 80 of 3 pence; and so, according to the definition of ratio, we may say that the ratio of the income to the tax is 80. On the other hand, when the two things connected by a rate (there being so much of the one per so much of the other) are of different kinds, there is no ratio between them. Thus we are not entitled to speak of the ratio of 9 pence to 1 yard of ribbon, because we cannot say that 9 pence is any numeric of 1 yard of ribbon; it is not $\frac{1}{5}$th of a yard of ribbon, nor 4 of a yard, nor $5\frac{3}{7}$ of a yard, nor any numeric, whole or fractional, of a yard of ribbon; and so there is no ratio between the two quantities.

The foregoing examples may suffice to aid the comprehension of the following more general description of the meaning of the word *rate*, and of *equivalence of rates*.

When there are two variable things of different kinds, or of the same kind but differently considered, which are connected by any condition making them mutually proportional (or, in other words, making either be proportional to the other), the relation between any value of the one and the corresponding value of the other is usually called a RATE; and the rate between any value of the one and the corresponding value of the other is *the same as*, or is *equivalent to*, the rate between any other value of the one and the corresponding value of the other.

The general description just given may now be further illus-

trated by a few additional examples of rates briefly stated for comparison.

Thus, for water issuing as an ornamental jet from a pipe, we may say that it flows at the rate of 12 gallons per minute. This is a rate between two quantities—so much water per so much time.

Again, we might say: *If there be five men to one boat, how many men would there be to seven boats at the same rate?* This is an example of the word rate as applied to cases in which so many things of one kind are connected with so many things of some other kind.

Further, we might have a rate between *a quantity of one kind of thing and a number of things of a different kind.* Thus we might say, at the rate of $\frac{3}{4}$ of a pound of beef per man, how much beef would be required for 20 men?

Also we may have, as before mentioned, in respect to income tax, so much tax per so much income. This is a rate between two things of the same kind—both money, but differently considered.

A statement of sameness or equivalence of two rates, their terms being arranged alike in order, may be called a RATE-EQUIVALENCE.

The sameness or equivalence of two rates having their terms arranged alike in order may be denoted by interposing between the two rates the sign $\fallingdotseq$, which may be called the sign of equivalence, and which may be noticed as having a resemblance to, but being different from, the sign of equality ($=$), used for denoting equality between quantities or numerics.* So a statement of rate-equivalence may be written thus:—

2 miles per 10 minutes $\fallingdotseq$ 12 miles per hour.

Often it occurs that while same rates differently expressed between two mutually connected things different in kind or in mode of consideration have to be dealt with, different rates between such things have also to be dealt with; and also it often occurs that the rate between two things mutually connected is subject to gradual change. Thus the rate between length of ribbon and money cost may vary from time to time for the same kind of ribbon; or, otherwise, the rate between length of ribbon and money cost, while remaining unchanged for different quantities of the same kind of ribbon, may be different for different kinds of ribbon. Also the rate between any very short time and the distance travelled in that time varies for a railway train leaving a station, and going forward, sometimes on a level, and sometimes up and sometimes down hill. Such a rate is commonly expressed as a velocity; and in this case of the railway train the velocity would be called a varying velocity.

* It would be an abuse of the sign of equality to employ it for expressing sameness or equivalence of rates in ordinary cases, such as those which have been brought forward above in the text. A different mode of expressing rates, which is often useful, especially in mathematical operations, will be explained further on, at pages 109 and 110; and under that mode, in which rates are treated as quantities admitting of increase and diminution, it can be made quite proper to express equivalence of rates by interposing the sign of equality.

Now, in dealing with rates that are subject to variation, it is often convenient to indicate the variation of the rate, or to express different rates, by stating the variable quantity of one of the two connected things which corresponds to an arbitrarily selected fixed quantity or unit of the other one of the two things. Thus we might state the different rates between water evaporated and coal consumed in different boilers by stating for these different boilers how much water is evaporated per pound of coal consumed. In this way we would be selecting a constant quantity of the coal, taken as a convenient unit, a pound, for one term of the rate, and treating the water evaporated for that fixed quantity as a variable term of the rate. The rate thus expressed, with the evaporation treated as variable, for a constant amount of consumption taken as a convenient unit of coal consumed, may be designated as the *rate of evaporation relative to consumption*. It might even, according to common usages, for brevity, frequently be spoken of as the *rate of evaporation*; the fixed term of the rate when taken as a unit being often left unmentioned, and being left to be understood through collateral explanations or through some customary usage. Thus the price of ribbon of a certain kind may often be stated as 1*s.* 3*d.*, meaning 1*s.* 3*d.* per yard; or the price of cotton as imported may be stated as 6¼*d.*, meaning 6¼*d.* per pound. The price in such cases is a rate between money and something sold; and in this mode of expression we may speak of a greater or a smaller rate, just as we speak of a higher or a lower price. Again, otherwise, the rate between water evaporated and coal consumed in the example previously considered might often conveniently be expressed by making the quantity of water for one term of the rate be constant and be a convenient unit, with, for the other term, a variable quantity of coal corresponding to that fixed quantity of water. Thus we might often find it convenient to specify, for a certain boiler, that the rate between evaporation and consumption is 1½ ℔. coal consumed per gallon of water evaporated; and, for another boiler, that the rate is 2¼ ℔s. coal per gallon of water. In this way, where the coal consumed is made the variable term in the statement of the rate, and the water evaporated is made constant, and is chosen so as to be a convenient unit-quantity (a gallon in this case), the rate would usually be designated as the *rate of consumption relative to evaporation*, or would sometimes briefly be called the *rate of consumption*, its relation to the evaporation of a gallon of water being left to be understood, if not formally mentioned.

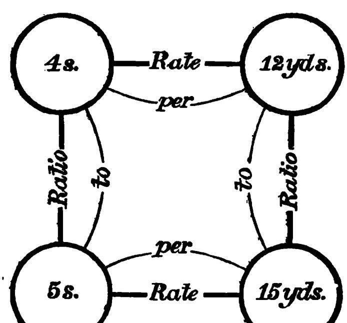
From the explanations just given, it may now readily be understood that when, as is often the case in practice, high rates and low rates, greater rates and less rates, are spoken of there is always one of the two terms of the rate distinctly stated, or implied, as being the one whose increase or diminution, relatively to the other taken as constant, constitutes the increase or diminution of the rate.

For the mere statement of a rate between two things, it is in principle a matter of indifference which of the two terms of the rate be named first. Thus we may express the same rate between time and distance travelled, according to convenience, as 5 minutes

per mile, or as 1 mile per 5 minutes. Or again, we may express the rate between coal consumed and water evaporated in a boiler, as being 2 gallons of water to 3 pounds of coal, or 3 pounds of coal to 2 gallons of water. When, however, it is wanted to treat a rate quantitatively (or, in other words, to regard it as a thing that can be increased or diminished—can be doubled or trebled, for instance) it is usual to state first the term with whose increase and diminution, relatively to the other taken as constant, the rate is chosen by definition as increasing and diminishing.

For example, in speaking of supplies of water for town use, or for water-power, or for other purposes, people often speak of the quantity of water flowing in a river or stream as being so many gallons per minute. Now the *quantity* thus spoken of is really a *rate*. It is a rate between water and time, and is expressed as so much water per unit of time. We can, then, obviously speak intelligibly of a double or treble rate of flow of water per minute; and this example may suffice to indicate what is meant when rates are treated as quantities and expressed numerically. Thus if we select 1 gallon per second as a unit rate of flow, twice that unit rate of flow would be 2 gallons per second; and a flow of 5 gallons per second would be five times that unit rate of flow; and so on.

The relations subsisting between *ratio-equation* and *rate-equivalence* may be very clearly illustrated by the accompanying diagram.



Here we see that between 4*s*. and 5*s*. there is a *ratio*, and between 12 yards and 15 yards there is a *ratio*, and that these two ratios are equal. The diagram thus exhibits the ratio-equation which may be written as follows:—

4*s*. : 5*s*. :: 12 yards : 15 yards;

or, Ratio 4*s*. to 5*s*. equal to ratio 12 yards to 15 yards.

Also in the same diagram we see that between 4*s*. and 12 yards there is a *rate*, and that between 5*s*. and 15 yards there is a *rate*,

and that these two rates are equivalent. The diagram thus also exhibits the rate-equivalence, which may be stated as follows:—

Rate 4*s.* per 12 yards, equivalent to rate 5*s.* per 15 yards;

or, 4*s.* per 12 yards $\doteq$ 5*s.* per 15 yards.

The diagram may also aid in explaining that while there is a rate between 4*s.* and 12 yards, and an equivalent rate between 5*s.* and 15 yards, but no ratio between the quantities in either of these pairs, because the quantities are of dissimilar kinds, yet there is a ratio between the numbers 4 and 12 occurring in the first pair, and there is a ratio between the numbers 5 and 15 occurring in the second pair, and these ratios are equal; * so that we may truly make the statement, as a proportion or ratio-equation, that

As $4 : 12 :: 5 : 15$;

while it would be nonsense to put forward, as a ratio-equation, the statement

As 4*s.* : 12 yards :: 5*s.* : 15 yards;

or to say, Ratio of 4*s.* to 12 yards = ratio of 5*s.* to 15 yards.

The principle exemplified in the last foregoing paragraph, and proved in its foot note, for cases in which either all the four numerics or the first three of them are integers, can easily be further proved to hold good in general, though any or all of the four numerics be fractional; and it may be stated in form of a proposition as follows:—

* That the two ratios of the numbers expressing the two equivalent rates in the present case, and that in general the two ratios of the numerics expressing the two equivalent rates in any like cases in which, for simplicity, we for the present restrict the first three to be "whole numbers," must be equal, may be shown by use of Prop. 3, page 97, and Prop. 4, page 103. Thus: if, in the example before us, we commence with the quantitative ratio-equation

4*s.* : 5*s.* :: 12 yards : 15 yards,

it follows, by Prop. 3, that the numerics, or numbers, expressing these quantities (the first pair both in one same denomination, and the second pair both in one same denomination) must form a ratio-equation. That numerical ratio-equation in the example before us is

$4 : 5 :: 12 : 15.$

Then from this it follows by Prop. 4, that the first will be to the third as the second is to the fourth; or, what is the same, that if the second and third numbers are interchanged, a new ratio-equation will so be formed by the four numerics taken in the altered order. That is, we have the four numbers in the case before us giving the new ratio-equation

$4 : 12 :: 5 : 15.$

If it were merely wanted to prove that the particular numbers 4, 12, 5, and 15 occurring in the present particular case are four proportionals when taken in this order, the proof might be given by mere arithmetic applied to these numbers themselves; but the proof which has been here offered is meant to show that the like holds good in general for any two equivalent rates. The proof in general might be further illustrated, or might perhaps be made clearer, by using four letters, such as a, b, c, and d, as general symbols, instead of the four particular numbers 4, 5, 12, and 15.

Proposition 7. In a rate-equivalence, if the first terms of the two rates be stated in the same unit (or denomination) each with the other; and if the second terms of the two rates, which may be different in kind from the first terms, be also stated in the same unit (or denomination) each with the other; the four numerics expressing these quantities in the order of their occurrence in the rate-equivalence will form a ratio-equation; or the ratio of the first to the second will be equal to the ratio of the third to the fourth.

Memorandum.—Although the learner may understand that this proposition holds good for fractional as well as for integral numerics, he should await its complete proof to his own mind before treating it as established for his use, except for cases in which the first three numerics are integers. Even under this limitation, however, the proposition is useful for many of the ordinary questions occurring in practice.

Rule II. When one rate is given, by two terms expressing it being given, and one term for an equivalent rate, or for what may be called the same rate in other terms, is given, and it is required to find the other term for that equivalent or same rate—(1.) Arrange the three given terms in a line in succession, so that when the required term shall be inserted in the fourth place the terms of the required rate shall be the second and fourth, and the terms of the given rate in corresponding order shall be the first and third. Observe that now the first and second terms will form a ratio, and that the third with the required fourth term are to form an equal ratio. (2.) Now, if the first and second terms be each stated in one denomination the same for both, and if the third term be in one denomination, find the product of the second and third terms regarded numerically,* and divide it by the first term regarded numerically. The result will be the required term in the same denomination as the third term. (3.) But if the first and second terms be not in one denomination the same for both, reduce them both to any one same denomination; and if the third term be in more denominations than

* The reason for saying *regarded numerically* is that we cannot multiply a quantity by a quantity, but we can multiply the number of units expressing one quantity by the number of units expressing another quantity. Thus we cannot multiply 15 yards by 4 shillings, nor 4 shillings by 15 yards, but we can multiply 15 by 4, or 4 by 15, and so find their product 60.

one, reduce it to one denomination; and then proceed as directed in (2).

Exam. 6. If 12 yards of ribbon cost 4 shillings, what will 15 yards cost at the same rate?

Here we have a rate given as 12 yards per 4 shillings; and, for an equivalent rate, or for the same rate otherwise expressed, we have the first term given as 15 yards, and we are to find the other term for that equivalent rate. According to the rule, the given term of the required rate, that is, 15 yards, is put in the second place, so that, the fourth place being reserved for the required term, which is here denoted by x shillings, the second and fourth will form the required rate. Then the other two terms forming the given rate, which are 12 yards and 4 shillings, are put in the first and third places, in the mutual order of rate corresponding to that of the second and fourth; so that 12 yards goes to the first place, and 4 shillings to the third place. Here we may name the given rate as 4 shillings per 12 yards, and the required rate as x shillings per 15 yards. Now, seeing that the first and second terms are each stated in one denomination the same for both, namely, yards, and that the third term is in one denomination, namely, shillings, we multiply the numbers in the second and third terms together, and divide the product by the number in the first term. The result of this operation is the number 5, which is the number expressing the required answer in the same denomination as that used in the third term, namely, shillings; and so the answer required is 5 shillings.

12 yds. : 15 yds. :: 4 shillings : x shillings.

$$\begin{array}{r} 4 \\ \hline 12)60 \\ \hline 5 = x. \end{array}$$

Hence,

12 yds. : 15 yds. :: 4 shillings : 5 shillings.

Or the work may be briefly written thus:—

$$\begin{array}{l} \text{yds.} \quad \text{yds.} \quad \textit{s.} \quad \textit{s.} \\ \;12 \;:\; 15 :: 4 : 5 \\ \qquad\quad 4 \\ \quad 12)60 \\ \qquad\quad 5\textit{s., answ.} \end{array}$$

Exam. 7. If 15 feet 8 inches of platinum wire cost £2 - 5 - 4, how much would 2 ft. 3 ins. of the same quality of wire cost at the same rate?

Here we have a rate given as 15 ft. 8 ins. per £2 - 5 - 4; and, for an equivalent rate (or for the same rate otherwise expressed), we have the first term given as 2 ft. 3 ins., and we are to find the other term for that equivalent rate. According to the rule, the given term of the required rate, that is, 2 ft. 3 ins., is put into the second place, so that, reserving the fourth place for the required term, which is here represented, as a quantity of money, by X,* we

* The capital letter X is here, for distinction, put to represent the *quantity of money* which will constitute the answer, irrespective of the units, whether shillings or pence, or partly shillings and partly pence, in which it may be expressed; while in the previous example the small letter x was put to denote the *number*, or *numeric generally*, which would express the answer in shillings.

shall have the second and fourth forming the required rate. Then the other two terms forming the given rate (which are 15 ft. 8 ins. and £2 - 5 - 4) are put in the first and third places in the mutual order of rate which corresponds to that of the second and fourth; so that 15 ft. 8 ins. goes to the first place, and £2 - 5 - 4 to the third place. (Here we may name the given rate as £2 - 5 - 4 per 15 ft. 8 ins., and the required rate as *X*, a sum of money, per 2 ft. 3 ins.)

ft. ins.	ft. ins.	£ s. d.	Money.
15 - 8 :	2 - 3 ::	2 - 5 - 4 :	*X*.
12	12	20	
188 ins.	27 ins.	45	
		12	
		544 pence	
		27	
		3808	
		1088	
		188)14688($78\frac{24}{188}$ pence,	*answ*;
		1316	
		1528	
		1504	or £0 - 6 - $6\frac{24}{188}$, *answ.*
		24	

We may now notice that the first and second terms constitute a ratio of length to length, and that the third and fourth are to constitute a ratio of money to money; and that the fourth term is to be found such that these two ratios shall be equal. Now, seeing that the first and second terms are not in one denomination the same for both, we reduce them to one denomination, inches; and so we find 188 ins. and 27 ins. to be used as altered expressions for the first and second terms. Also, seeing that the third term is in more denominations than one, we reduce it to one denomination, pence, and so obtain 544 pence as an altered expression for the third term. Then, according to the rule, we multiply the numbers in the second and third terms together (that is, we multiply together 544 and 27), and we divide their product, 14688, by the number in the first term, that is, by 188; and so the fourth term required is found to be $78\frac{24}{188}$ pence; * or we may say accurately enough for most purposes 78 pence, since $\frac{24}{188}d.$ is obviously less than a farthing; and this by reduction is found to be £0 - 6 - $6\frac{24}{188}$, or £0 - 6 - 6 approximately.

Rule III. The same being given and the same required as in Rule II., the mode of procedure may as well be, and sometimes may more conveniently or more obviously be, as follows:—(1.) In a line of four successive places put the pair of terms of the given rate into the first and second places, making the one which corresponds in kind or in mode of consideration with the required term stand second; and put the given term

* The remainder 24, at the close of the division by 188, indicates that, besides the 78 found in the proper "quotient" as a number of pence, there is also the fraction $\frac{24}{188}$ of a penny required to make up the complete answer with perfect exactitude. Explanations on this subject have been already given in the chapters on Division and in the Introductory Chapter on Fractions. Further instructions will be found in Rules IV. and V. in the present chapter on Proportion.

of the required rate into the third place, and reserve the fourth place for the required term. (2.) Now if the first and third terms of the four, or, what is the same, if the first terms of the two pairs of equivalent rates be each stated in one denomination the same for both, and if also the second term of the first pair be in one denomination, find the product of the second and third terms, regarded numerically, and divide it by the first, regarded numerically. The result will be the required term in the same denomination as the second term. (3.) But if the first terms of the two pairs of equivalent rates be not in one denomination the same for both, reduce them both to any one same denomination, and if the second term of the first pair be in more denominations than one, reduce it to one denomination; and then proceed as directed in (2).

Exam. 8. If 2 cwt. 3 qrs. 17 ℔s. of sugar cost £4 - 5 - 9, what is the value of 15 cwt. 1 qr. 14 ℔s. at the same rate?

Here the given rate is 2 cwt. 3 qrs. 17 ℔s. for £4 - 5 - 9; and, according to the rule, these are put into the first and second places, the £4 - 5 - 9 being made to stand second, as it is the one which corresponds with the required term; also the given term of the required rate, namely, 15 cwt. 1 qr. 14 ℔s., is put into the third place, while the fourth place is reserved for the required term, which, considered here as a quantity of money, may be represented by X. Now, seeing that the first terms of the two pairs of equivalent rates are not stated each in one denomination, the same for both, we reduce them both to one same denomination, which is conveniently made to be the lowest occurring in them, namely, pounds. Thus we get for these terms 325 ℔s. and 1722 ℔s. Also, seeing that the second term of the first pair, namely, £4 - 5 - 9, is in more denominations than one, we reduce it to one denomination, which is conveniently made to be the lowest

```
cwt. q. ℔s.      £ s. d.   cwt. q. ℔s.  Money.
 2 - 3 - 17  per 4 - 5 - 9 :: 15 - 1 - 14 per X.
 4               20            4
 --              --           --
 11              85           61
 28              12           28
 ---            ----          ---
 105            1029 pence    502
 22             1722          122
 ---            ----         ----
 325 ℔s.        2058         1722 ℔s.
               2058
              7203
             1029
          ---------
       325)1771938(5452 38/325 pence
           1625
           ----
            1469
            1300            12)5452 38/325
            ----            ----------
             1693          2,0)45,4 - 4
             1625            ----------
             ----            £22 - 14 - 4 38/325
              688
              650          or approximately
              ---
               38            £22 - 14 - 4, answ.
```

occurring in it; and so we get for this term 1029 pence. Thus we obtain

325 lbs. per 1029 pence $\asymp$ 1722 lbs. per X.

In this we next have to multiply the numbers in the second and third terms together, and to divide the product by the number in the first. The result is $5452\frac{38}{325}$, which expresses the required term in the same denomination as the second term, that is, in pence. This, for convenience, may be reduced, as shown in the margin, to pounds, shillings, and pence;* and so may be expressed as £22 - 14 - $4\frac{38}{325}$, or as £22 - 14 - 4 approximately.

Remark.—The two varied methods of procedure taught in Rules II. and III., with the explanations of principles as to ratio and rate which precede them, and the illustrations afforded in the examples attached to them, may clear up, amend, and thence reconcile two discrepant views as to allowable modes of thought and of practical operation, which have prevailed very much among different writers on arithmetic, and in the practice of different men in business, in respect to questions in Proportion. It was long the prevailing usage to place the terms of questions in proportion in the order indicated by the following example:—

As 4 shillings : 12 yards :: 5 shillings : 15 yards.

By others, and especially by the more scientific authors in later times, this method was strongly objected to, on the ground of its being dealt with as involving the absurdity of there being ratios between quantities which are heterogeneous; as in the present case the absurdity of an assumption of a ratio between 4 shillings and 12 yards, and an equal ratio between 5 shillings and 15 yards. Such writers, in amending the treatment of the subject, have rightly taught that no such ratios between quantities unlike in kind have any existence; and, assuming that a statement of equality of two ratios is essentially wanted, they have taught that the statement to be used, and to be called a proportion, ought to be made thus:—

As 4 shillings : 5 shillings :: 12 yards : 15 yards.

The explanations given above, in the present treatise, show that both modes of arrangement admit of being thought out scientifically, and employed practically, perfectly well; if in the one case we regard the statement as a *rate-equivalence*, and in the other as a *ratio-equation*. The actual arithmetical work comes to be quite the same, whichever arrangement of the terms and whichever true mode of thought be adopted.

RULE IV. If at the close of the numerical division by the first term directed in Rules II. and III., or in other cases of the same kind, there be a remainder, that

* Reduction from a lower to one or more higher denominations, such as is here explained for this particular example, will be found prescribed for ordinary use further on, in Rule VI., page 117, and explained further in Example 11.

remainder may be put as a numerator with the divisor as a denominator for a fraction to be annexed to the proper or integral quotient already obtained in the division, so as to obtain the complete answer expressed by a fractional numeric in the one denomination in which the answer is to come out under Rules II. and III.

RULE V. But when the required result is to be a quantity of a kind which is usually expressible in more denominations than one, if the result has not at once come out in the lowest convenient denomination *without remainder*, it is usually requisite to reduce the remainder to a lower denomination, and to continue the operation as in compound division, so as to avoid fractional expressions unless in the lowest denomination.

RULE VI. When the result comes out in a denomination from which it can be wholly or partly reduced to one or more higher denominations, it is usually requisite or desirable for convenience to make such reduction.

NOTE.—*The following examples* (9, 10, *and* 11) *are in illustration of Rules IV., V., and VI.*

Exam. 9. If a web of linen containing 26 yards cost £3, what would 732 yards cost at the same rate?

Here the terms are arranged for the work in the order used in Rule II. Then the second and third terms, regarded numerically, are multiplied together, and their product, 2196, is divided by 26. The numerical result of this division, whether integral or fractional (or, in other words, whether a proper number or a fractional numeric), will express the answer in the denomination of the third term, that is here in pounds sterling. Then seeing that, in dividing by 26, we get the integral quotient 84 and the remainder 12, we see, according to the principles and rules of Division, that the 26th part of 2196 is $84\frac{12}{26}$. Thus the result is $84\frac{12}{26}$, or 84 pounds and $\frac{12}{26}$ of another pound sterling.* It is, however, usually not practically convenient to express a quantity of money in any such fractional form as $\frac{12}{26}$ of a pound; and any such fractions of any denomination above the lowest commonly in use,

```
   yds.  yds.   £
As 26 : 732 :: 3 :
             3
     26)2196(£84 12/26
        208
        ---
         116
         104
         ---
          12 remainder.
```

* If any clearing up of the reason for this interpretation of the fractional numerical result $84\frac{12}{26}$ is wanted, it will be obtained in the foot note next following.

or conveniently available, are ordinarily to be reduced to equivalent expressions in one or more lower denominations, as is directed in Rule V., and as is illustrated in the following example.

Exam. 10. *Same question as in Exam.* 9.

Here, after we have found £84 as the first part of the answer, there is a numerical remainder 12, and this we have a right to treat as meaning £12,* of which one twenty-sixth part is to be found and to be added to the £84, to make up the total required result. Then the £12 is reduced to shillings, and the operation proceeds as in compound division. At the end there is a remainder of 2 farthings, and, for perfect exactitude, $\frac{1}{26}$th of this ought to be added to the approximate result, £84 - 9 - $2\frac{3}{4}$, already found; but this fraction of a farthing, being too small a quantity of money to be practically used, is rejected.

```
   yds. yds.   £
As 26 : 732 :: 3 :
              3  £   s.  d.
    26)2196(84 - 9 - 2¾
       208
       ---
        116
        104
        ---
         12 pounds remainder at this stage.
         20
        ---
        240
        234
        ---
          6 shillings remainder at this stage.
         12
        ---
         72
         52
        ---
         20 pence remainder at this stage.
          4
        ---
         80
         78
        ---
          2 farthings remainder at the end.
```

Exam. 11. *This example is for illustration of Rule VI.; and the same question is taken as in Example* 8. If 2 cwt. 3 qrs. 17 lbs. of sugar cost £4 - 5 - 9, what is the value of 15 cwt. 1 qr. 14 lbs. at the same rate?

The work for this question was carried out in Example 8 by commencing with a statement of rate-equivalence; but here, for variety, it is opened with a statement of ratio-equation. After reduction of the two stated weights of sugar to lbs., and of the stated sum of money to pence, if we put the small letter x to denote the

* The best mode for clearly explaining the reason why this numerical remainder, 12, is taken as meaning £12, is by going back to the original definition of ratio-equation, and using the definition as offered in a second mode of expression, on page 95. By that we see that what numeric 732 yards is of 26 yards, the same numeric must the required consequent in the second ratio be of its antecedent, £3. But 732 yards is $\frac{732}{26}$ of 26 yards; and so the required quantity of money must be $\frac{732}{26}$ of £3: or it must be $\frac{1}{26}$th of 732 times £3; or it must be $\frac{1}{26}$th of £2196; and this, as the work proceeds, comes out to be £84, together with $\frac{1}{26}$th of £12.

$$\begin{array}{lllllllll}
\text{cwt.} & \text{q.} & \text{lbs.} & \text{cwt.} & \text{q.} & \text{lbs.} & £ & s. & d. \ \text{Money.} \\
2 - & 3 - & 17 : & 15 - & 1 - & 14 :: & 4 - & 5 - & 9 : X
\end{array}$$

$$\begin{array}{lll}
\underline{4} & \underline{4} & \underline{20} \\
11 & 61 & 85 \\
\underline{28} & \underline{28} & \underline{12} \\
105 & 502 & 1029 \text{ pence.} \\
\underline{22} & \underline{122} & \\
325 \text{ lbs.} & 1722 \text{ lbs.} & \\
 & \underline{1029} & \\
 & 15498 & \\
 & 3444 & \\
 & \underline{1722} &
\end{array}$$

$$\begin{array}{ll}
325)1771938(5452\tfrac{38}{325} \text{ pence.} & \\
\underline{1625} & \\
1469 & 12)5452\tfrac{38}{325} \\
\underline{1300} & 2,0)45,4 - 4 \\
1693 & £22 - 14 - 4\tfrac{38}{325}; \\
\underline{1625} & \\
688 & \text{or approximately} \\
\underline{650} & £22 - 14 - 4, \textit{answ.} \\
38 &
\end{array}$$

numerical expression for the answer in pence, we have the ratio-equation—

Ratio 325 lbs. : 1722 lbs. = ratio 1029 pence : x pence;

and consequently we have the numerical ratio-equation—

Ratio 325 : 1722 = ratio 1029 : x.

Working this out, we find x to be the fractional numeric $5452\frac{38}{325}$, which signifies that the answer is 5452 pence, together with the fraction $\frac{38}{325}$ of another penny. This fraction of a penny is obviously so small a quantity of money (being, indeed, less than half a farthing) that it may be neglected, and the required quantity of money may be taken as 5452 pence. This then, for convenience, according to Rule VI., is reduced to its equivalent expression by use of larger denominations, and so the answer is found as £22 - 14 - 4.

The modes of thought and of procedure put forward in Rules II. and III. have involved the making out of a numerical ratio-equation, and have depended on, or have been connected with, the principle that in a numerical ratio-equation the product of the extremes is equal to the product of the means. This principle is very usually taken as the guide to the modes of procedure most commonly taught in treatises on arithmetic. A better mode of thought—a mode which is simpler and clearer—is presented to the mind by the next following rule, which, it will be seen, results directly from the definition No. 2 of ratio-equation, given on page 95.

Rule VII. If, for a ratio-equation, one of the two equal ratios is given, and the antecedent for the other ratio is given, while the consequent for that ratio is required—to find the required consequent, take the same numeric of its antecedent as the given consequent is of its antecedent.

Or, in other words: If the first three terms of a ratio-equation are given, and the fourth term is required—take, for the fourth term, the same numeric of the third that the second is of the first.

Exam. 12. *Same question as in Exam.* 6. If 12 yards of ribbon cost 4 shillings, what will 15 yards cost at the same rate?

Here, obviously, after writing down the ratio-equation according to the instructions in Rule II., part (1), we may proceed in the following way:—The required money is to be the same numeric of 4 shillings that 15 yards is of 12 yards. But 15 yards is $\frac{15}{12}$ of 12 yards. Hence x shillings must be $\frac{15}{12}$ of 4 shillings. That is to say, the required money will be found by taking $\frac{1}{12}$ of 15 times 4 shillings; and so the work comes to be as in the margin, where 15 times 4 shillings is 60 shillings, but where, however, in respect to form of procedure, 15 is multiplied by 4, instead of 4 by 15; because the same result is brought out a little more easily by taking the smaller number as the multiplier with the greater as the multiplicand.

$$12 \text{ yards} : 15 \text{ yards} :: 4 \text{ shillings} : x \text{ shillings.}$$
$$12 : 15 :: 4 : x$$
$$\underline{4}$$
$$12)\underline{60} \text{ shillings.}$$
$$5 = x$$

Answer, 5 shillings.

It is to be noticed that the varied mode of consideration taught here leads to the same arithmetical work as did that on which the process in Rule II. was founded, and which depended on or was connected with, the principle that, in a numerical ratio-equation, the product of the extremes is equal to the product of the means.

Exam. 13. *Same question as in Exam.* 7. If 15 feet 8 inches of platinum wire cost £2 - 5 - 4, how much would 2 feet 3 inches of the same quality of wire cost at the same rate?

Here, after writing down the ratio-equation, as was done in Exam. 7, under the directions given in Rule II., part (1), we proceed to find what numeric the consequent in the first ratio is of its antecedent. To do this we reduce them both to one same denomination, and so find for the antecedent and consequent 188 ins. and 27 ins. Hence obviously the consequent is $\frac{27}{188}$ of the antecedent. Hence X, the consequent for the required ratio, must be $\frac{27}{188}$ of its antecedent. That is to say, the required term must be $\frac{27}{188}$ of £2 - 5 - 4. Now this required money is most readily found by

reducing the £2 - 5 - 4 to pence, and then multiplying that sum in pence by 27, and dividing by 188; whereby the result is found to be $78\frac{24}{188}$ pence, or £0 - 6 - $6\frac{24}{188}$, or £0 - 6 - 6 approximately. Thus, by the present mode of thought, without introducing the principle that, in a numerical ratio-equation, the product of the extremes is equal to the product of the means, we are led into the same arithmetical work, and to the same result, as by the instructions in Rule II., depending on or usually associated with that principle.

$$
\begin{array}{llr}
\text{ft. ins.} & \text{ft. ins.} & \text{£ \ s. \ d. \ Money.} \\
15 - 8 \ : & 2 - 3 \ :: & 2 - 5 - 4 \ : \ X. \\
\underline{12} & \underline{12} & \underline{20} \\
188 \text{ ins.} & 27 \text{ ins.} & 45 \\
 & & \underline{12} \\
 & & 544 \\
 & & \underline{27} \\
 & & 3808 \\
 & & \underline{1088} \\
 & & 188)14688(78\frac{24}{188} \text{ pence,} \\
 & & \underline{1316} \\
 & & 1528 \quad \text{or £0 - 6 - 6, } \textit{answ.,} \\
 & & \underline{1504} \quad \textit{approximately.} \\
 & & 24
\end{array}
$$

RULE VIII. When for a ratio-equation, for which the fourth term is required, the third term is of more denominations than one, if the given ratio, either as given originally or as subsequently modified by reduction of both its terms to one same denomination, is expressed as a ratio of two numbers, and if neither of these is greater than 12, it is often an easy mode of procedure not to reduce the third term to one denomination; but, retaining it as a quantity expressed in various denominations, to multiply it by the number which constitutes the second term, and to divide the result by the number which constitutes the first term.

Note.—The procedure here directed may be understood as being a case of taking for the fourth term the same numeric of the third that the second term is of the first.

Exam. 14. If nine sheep cost £20 - 10 - 7½, what is the value of 5 of them at the same rate?

Here, after writing down the first three terms of the ratio-equation, as usual, we may proceed by noticing that the second term is $\frac{5}{9}$ of the first, and that therefore the fourth term must be $\frac{5}{9}$ of the third. Hence, to find the fourth, we take $\frac{5}{9}$ of £20 - 10 - 7½, and so we get for the answer £11 - 8 - 1½.

$$
\begin{array}{r}
9 \text{ sheep} : 5 \text{ sheep} :: \text{£}20 - 10 - 7\frac{1}{2} : \\
\underline{5} \\
9)102 - 13 - 1\frac{1}{2} \\
\text{£}11 - 8 - 1\frac{1}{2}, \textit{answ.}
\end{array}
$$

RULE IX. For finding the fourth term for a ratio-equation, the work may often be much abbreviated by dividing the first and second, or the first and third terms, by any number which divides each of the two without remainder.

The allowableness of this abbreviation may be readily perceived through noticing that in the work, if performed without the abbreviation, we have to carry out a process of division in which, after any necessary reductions of terms to suitable denominations, the product of the second and third terms is the dividend, and the first term is the divisor; and that if we divide the dividend and divisor by any number we shall get a new dividend and divisor, which will give the same result as the previous pair would have done; and, further, that by dividing either the second or the third term by any number before taking their product, we get the same as if we took their product first and divided by that number afterwards.

Otherwise: The principle on which the allowableness to divide the first and second terms by any number depends, may well be viewed as included in Proposition 6, page 104, which was, that *a ratio is not altered by multiplying or dividing both its terms by the same number*. Also, the principle on which the abbreviation by dividing the first and third terms by any number that will divide them both without remainder depends, may be viewed as being comprised in the following two propositions.

PROPOSITION 8. If the antecedent in a ratio is multiplied or divided by any number, the ratio is thereby multiplied or divided by the same number.

PROPOSITION 9. Hence: If the antecedents of two equal ratios are multiplied or divided by the same number, the consequents being retained unchanged, the altered ratios resulting will be equal.

Thus, for example, if we begin with the ratio-equation

12 yards : 16 yards :: 21 shillings : 28 shillings,

and if we divide the antecedents in both ratios by 3, we get—

$\frac{1}{3}$ of 12 yards : 16 yards :: $\frac{1}{3}$ of 21 shillings : 28 shillings;

or, 4 yards : 16 yards :: 7 shillings : 28 shillings.

So, obviously, if the fourth term, 28 shillings, were not given, but were required to be found, it would come out the same whether we were to work under the original ratio-equation, or under the altered one in which the ratios are altered each to $\frac{1}{3}$rd of its original numerical value, but are still mutually equal.

Exam. 15. If 15 yards of linen cost £2 - 1 - 10$\frac{1}{2}$, what will 55 yards cost at the same rate?

This question is here worked out under the methods offered in Rules IX. and VIII.

The first and second terms, being noticed to be divisible by 5, are divided by that number, and a new pair of terms, 3 and 11, are obtained, which have the same ratio and are used instead of them. Then, as is taught in connection with Rule VIII., the answer must be $\frac{11}{3}$ of £2 - 1 - $10\frac{1}{2}$, because the second term is $\frac{11}{3}$ of the first term, as is seen by looking to the new pair of terms, 3 and 11, got for the first ratio.

	yds.		yds.		£ s. d.
As	15	:	55	::	2 - 1 - $10\frac{1}{2}$
	3	:	11		11
					3)23 - 0 - $7\frac{1}{2}$
					7 - 13 - $6\frac{1}{2}$, *answ.*

RULE X. When for a ratio-equation in which the fourth term is required, the two terms of the first ratio are expressed in one same denomination, and the third term is expressed in one denomination, if a fraction occurs in the expression of any one or more of the three given terms, it is often convenient to get an equivalent ratio-equation, clear of fractions, by multiplying either the first and second, or the first and third, by any number which will clear away one or both of the fractions which may occur in the pair of terms selected for multiplication, and by repeating, if necessary, the like process till all the fractions are cleared away. This procedure will alter, in accordance with Propositions 6 and 9 (pages 104 and 122), the original ratio-equation to a new one, which will have the fourth term unchanged. To multiply two terms, the first and second, or the first and third, by the denominator of a fraction occurring in either of them, will always clear away that fraction, and may sometimes clear away two fractions, one in each of that pair of terms: and it is generally convenient to multiply no further than is seen to be sufficient. Then the new ratio-equation so found is to be used instead of the original one, and its fourth term is to be found as usual, and will be the required fourth term for the original ratio-equation.

Exam. 16. If $3\frac{1}{2}$ gallons of oil weigh $33\frac{1}{4}$ lbs., what would $6\frac{1}{3}$ gallons of the same kind of oil weigh; (or what would $6\frac{1}{3}$ gallons weigh at the same rate between measure and weight)?

Here we may set down the ratio-equation—

gal.		gal.		lbs.		lbs.
$3\frac{1}{2}$	:	$6\frac{1}{3}$	::	$33\frac{1}{4}$	:	x.

In this we may multiply the first and third terms by 4, the denominator of the fraction in the third term, noticing that this process will obviously clear away the fractions in both these terms. Thus we get—

$$\begin{array}{cccc} \text{gal.} & \text{gal.} & \text{lbs.} & \text{lbs.} \\ 14 & : 6\frac{1}{3} :: & 133 & : x. \end{array}$$

Further, we may multiply the first and second terms of this new ratio-equation by 3, the denominator of the fraction in its second term, and so we get—

$$\begin{array}{cccc} \text{gal.} & \text{gal.} & \text{lbs.} & \text{lbs.} \\ 42 & : 19 :: & 133 & : x. \end{array}$$

And so x will be found from the numerical ratio-equation—

$$42 : 19 :: 133 : x.$$

This, by the usual process, gives $x = 60\frac{1}{6}$; and so the required answer is that $6\frac{1}{3}$ gallons of the oil would weigh $60\frac{1}{6}$ lbs.

It is to be noticed that after having once set down the original ratio-equation—

$$\begin{array}{cccc} \text{gal.} & \text{gal.} & \text{lbs.} & \text{lbs.} \\ 3\frac{1}{2} & : 6\frac{1}{3} :: & 33\frac{1}{4} & : x \end{array}$$

we might have, for brevity, dropped the designations of gallons and lbs., and have worked merely with numerical ratio-equations, going through exactly the same arithmetical processes, and bringing out the same result as the value of x, which must be recollected as being the numerical value of the answer in lbs.

Explanations on Proportionality of one Variable to another in cases when either or each of the variables is not continuously variable.

There are some kinds of questions which are often treated as falling within the scope of the subject of Proportion, but in which, on account of one or more of the variables being not continuously variable, there is often no perfectly correct answer possible to the question proposed; while to some other and very similar questions there may be perfectly true answers available. A few examples may serve sufficiently to indicate the general characters of such cases as are here referred to.

(1.) A question may be asked—If 12 eggs are sold for 14 pence, how many eggs may be bought for 3 shillings at the same rate?

Here, putting x to denote the number required of eggs, if there be a rate-equivalence, and consequently a ratio-equation, in this case possible, we shall have—

$$14 \text{ pence} : 36 \text{ pence} :: 12 \text{ eggs} : x \text{ eggs};$$

and so x will be found by the numerical ratio-equation—

$$14 : 36 :: 12 : x.$$

This, by the usual process, gives $x = \frac{36}{14}$ths of 12, or $x = 30\frac{6}{7}$. But

as we cannot practically, nor even strictly in principle, have $\frac{6}{7}$ths of an egg, there is no perfectly true answer possible to the question proposed; or, in other words, if we buy any number of eggs whatever for 36 pence, they will not in any case be bought at the same rate as that of 12 eggs for 14 pence. A true interpretation of the arithmetical result would be to say, that at the rate proposed of 14 pence per dozen eggs, the sum of money proposed would buy 30 eggs, and would comprise in addition $\frac{6}{7}$ths of the cost of one egg; or, what is the same, that the sum of money proposed would comprise 30 times the cost of one egg together with $\frac{6}{7}$ths of the cost of an egg.

Various other questions having like characters might be adduced, in which the answer is asked to be given as a number of some indivisible unit—a number of men, or of sheep, or of watches, for instance—and yet in which the arithmetical process, carried on as usual, brings out, not a number, but a fractional numeric. In such cases we must conclude that the question does not admit of a perfectly correct answer being given to it; but often a useful and perfectly true modified interpretation may be given to the fractional result.

(2.) If 5 men can carry a mast weighing 6 cwt., how many men would be required to carry another mast weighing 11 cwt. at the same rate between men and load?

Here, if there is any perfectly true answer possible, it will be obtained through the ratio-equation—

$$\text{Ratio } 6 \text{ cwt.} : 11 \text{ cwt.} = \text{ratio } 5 \text{ men} : x \text{ men};$$

whence we would have the numerical ratio-equation—

$$6 : 11 :: 5 : x,$$

which gives

$$x = 9\tfrac{1}{6}.$$

Now, as we cannot have $9\frac{1}{6}$ men, there is no true answer to the question proposed. We may, however, interpret the result otherwise by saying that the calculation shows that the whole weight of the mast comprises 9 times the weight allowed for one man, together with $\frac{1}{6}$th of the weight allowed for one man.

Thus we see that instead of the original statement for a ratio-equation we might properly have noted—

$$\text{As } 6 \text{ cwt.} : 11 \text{ cwt.} :: 5 \text{ loads for a man} : x \text{ loads for a man},$$

where x, instead of denoting a number of men, would denote a number of loads, each suited for one man, together with a fraction of another like load.

(3.) If 4 horses are worth 9 cows, how many horses would be value for 15 cows at the same rate?

Here, by usual methods, if we put x to denote the number required of horses, we would make the statement—

$$\text{As } 9 \text{ cows} : 15 \text{ cows} :: 4 \text{ horses} : x \text{ horses};$$

whence we would find

$$x = \tfrac{15}{9}\text{ths of } 4, \text{ or } x = 6\tfrac{2}{3}.$$

Thus the answer would be $6\frac{2}{3}$ horses. But, as we cannot have $\frac{2}{3}$ of a horse, we see that there is not any number of horses that would be value for 15 cows at the rate of 4 horses per 9 cows. We might interpret the arithmetical result by saying that 6 horses, together with $\frac{2}{3}$ of the value of another horse, would be value for the 15 cows.

Exercises.

31. If £14 - 17 - 6 pays for 10 cwt. 2 q. 14 lbs. of sugar, what would 4 cwt. 1 q. 14 lbs. cost at the same rate? *Answ.* £6 - 2 - 6.

32. If 18 cwt. 3 q. 21 lbs. of beef cost £72, what would 44 cwt. 0 q. 21 lbs. cost at the same rate? *Answ.* £168.

33. If a person walk 17 miles in 5 hours 12 minutes 31 seconds, how far would he walk at the same rate in 3 hours 40 minutes 36 seconds? *Answ.* 12 miles.

34. If the earth move 69,000 miles in its orbit in an hour, through what space does it move at the same rate in 16 minutes 48 seconds? *Answ.* 19,320 miles.

35. If the earth goes at the rate stated in the last preceding exercise, how far does it move in 22 minutes 38 seconds? *Answ.* $26,028\frac{1}{3}$ miles.

36. If it goes at the rate stated in the last two preceding exercises, what length of path does it traverse in a second?

A good mode of commencing the work for this question may be by noting a ratio-equation thus:—

$$1 \text{ hour} : 1 \text{ second} :: 69{,}000 \text{ miles} : x \text{ miles},$$

where x denotes the number of miles per second [or, more correctly, the numeric of mile per second].

Another way, also good, for solving the question may be by thinking on the case through the methods of Division, and without introducing any consideration of a ratio-equation. Thus: The distance passed over per minute must be $\frac{1}{60}$th of the distance per hour; and, further, the distance per second must be $\frac{1}{60}$th of the distance per minute.

Answ. $19\frac{1}{6}$ miles.

37. If 96 men reap 40 acres of grain in a week, how many would reap 65 acres in the same time, at the same rate between number of men and quantity of land reaped in a week? *Answ.* 156 men.

38. If 93 men reap 38 acres in a week, how many would reap 56 acres in the same time at the same rate between number of men and quantity of land reaped in a week? (See explanations for such questions on pages 124 and 125.) *Answ.* $137\frac{1}{19}$ men; or a more proper answer would be that the work of reaping 56 acres would be the work of 137 men for a week, together with $\frac{1}{19}$th of the work of one man for a week.

39. How much wheat at 10*s.* 9*d.* per cwt. may be purchased for £149 - 16 - 4? *Answ.* 278 c. 2 q. $25\frac{79}{129}$ lbs.

40. How much oats at 8*s.* 8*d.* per cwt. may be bought for £64 - 3 - $2\frac{1}{2}$? *Answ.* 148 c. 0 q. 7 lbs.

41. If 6 cwt. 3 q. 12 lbs. of flour cost £5 - 9 - 9, what would 4 cwt. 2 q. cost at the same rate? *Answ.* £3 - 12 - $0\frac{9}{32}$.

42. If 6 cwt. 3 q. 12 lbs. of flour cost £5 - 9 - 9, what is the cost per cwt.? *Answ.* 16*s.* $0\frac{1}{16}$*d.* per cwt.

43. If wheat is sold at 28*s.* 8*d.* per barrel of 20 stone, what is its cost per cwt.?

An easy mode of proceeding to bring out a solution here may be by noticing that we have the proportion, As 20 stone is to 1 cwt., so is the cost of the twenty stone to the cost of the hundredweight. But 1 cwt. is equal to 8 stone, and so we have the proportion—

As 20 stone : 8 stone :: 28*s.* 8*d.* : cost per cwt.

Answ. 11*s.* $5\frac{1}{2}$*d.* per cwt., approx.

44. If 39 c. 1 q. 14 lbs. of pork cost £88 - 1 - 5, what would 13 c. 2 q. 5 lbs. cost? *Answ.* £30 - 5 - 11, approx.; or £30 - 5 - $10\frac{4129}{4410}$, exactly.

45. At the rate stated in the foregoing question, what is the cost of the pork per cwt.?

Answ. £2 - 4 - 9 approximately, or £2 - 4 - $8\frac{1792}{2205}$ exactly.

46. Three gallons of water taken from the supply pipes of a town have been found to contain 47 grains of carbonate of lime in solution. How much carbonate of lime would there be in the daily supply to the town—7,542,000 gallons—at the same rate? *Answ.* 118,158,000 grains, or $16,879\frac{5}{7}$ lbs., or 7 t. 10 c. 2 q. $23\frac{5}{7}$ lbs.

47. If 2 gallons and 3 quarts of strong sulphuric acid weigh $50\frac{3}{4}$ lbs., how much will 27 gallons weigh at the same rate? *Answ.* 498 lbs. $4\frac{4}{11}$ oz.

48. If in painting 57 square yards of wall surface 26 lbs. of paint, mixed ready for the brush, have been used, how much paint will be required for 105 square yards at the same rate? *Answ.* $47\frac{51}{57}$ lbs., or approximately 48 lbs.

49. If in painting 254 square feet of wall surface $13\frac{1}{2}$ lbs. of paint, mixed for the brush, have been used, how much surface will be painted by $73\frac{3}{4}$ lbs. at the same rate?

Aid may be obtained for this from Rule X., page 123.

Answ. $1387\frac{16}{27}$ square feet.

50. Taking the ratio of the circumference of a circle to the diameter as being very approximately equal to the ratio of 113 to 355, find the equatorial circumference of the earth, regarding it as the circumference of a circle 7926 miles in diameter. *Answ.* $24,900\frac{39}{113}$ miles.

51. The earth revolves round the sun in an orbit, which is very nearly circular, and has its diameter about 182,900,000 miles. What is the length of the circumference of a circle having that diameter? *Answ.* 574,600,000 miles, approximately.

52. A cubic foot of water weighs 1000 ounces avoirdupois, nearly. What is the weight of the water contained in a vessel, if the bulk is found by measurement to be $217\frac{1}{2}$ cubic inches? *Answ.* 7 lbs. $13\frac{125}{144}$ oz.

53. The earth describes its orbit round the sun in 365 days, 6 hours, 9 minutes, and 10 seconds; through what space does it move each hour, at an average, the circumference of the orbit being 574,000,000 miles? *Answ.* $65,479\frac{389615}{3155815}$ miles.

54. At 7*s.* 6*d.* per ounce, what is the value of a silver bowl weighing 9 oz. 13 dwts. 8 grs.? *Answ.* £3 - 12 - 6.

55. If a clerk has a salary of £89 - 12 - 6 per year, commencing on the morning of the 1st of May, how much has he to receive on leaving his situation on the evening of the 17th of December, the first and last days stated being included as days of his service, and the year being reckoned as 365 days? *Answ.* £56 - 14 - $5\frac{13}{73}$.

56. If the rent of 59 a. 3 r. 20 p. is £134 - 4 - 0, what would be the rent of 12 a. 2 r. 30 p. at the same rate? *Answ.* £28 - 8 - $8\frac{416}{479}$.

57. What would be the cost of 365 bottles of wine, at £2 - 13 - 6 per dozen? *Answ.* £81 - 7 - $3\frac{1}{2}$.

58. What is the cost of 311 sheep, at £37 - 12 - 6 per score? *Answ.* £585 - 1 - $4\frac{1}{2}$.

MEASURES AND MULTIPLES.

When one number is contained in another any number of times *exactly*, the greater is called a MULTIPLE of the less; and the less is said to be a MEASURE of the greater, or to MEASURE it.* Also any number is said to be a MULTIPLE of itself, and any number is said to be a MEASURE of itself, or to MEASURE itself.

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* Also when one number, or numeric generally, is contained in another any number of times exactly, the greater is often spoken of as being *divisible* by the less, this expression being an abbreviation for *divisible without remainder or fraction*, or for *divisible equably and integrally*, and the less is sometimes called a *submultiple* or an *aliquot-part* of the greater. Likewise, any number, or numeric generally, is said to be divisible by itself. Thus 18 is *divisible* by 2, 3, $4\frac{1}{2}$, and 6; and thus also 2, 3, $4\frac{1}{2}$, and 6, are called measures, submultiples, or aliquot-parts of 18, being contained in it 9, 6, 4, and 3 times. Also 18 is *divisible* by 18; and 18 is called a *measure* of 18. Also the fractionally expressed quantity $\frac{3}{4}$*d.* is called an aliquot-part of 6*d.*, being $\frac{1}{8}$th of that money. The designation *aliquot part* is little used except in the method of calculation called "Practice," which is much employed in mercantile reckoning, and which will be found taught further on in this treatise.

For expressing the same meaning as is conveyed by *divisible equably and integrally*, or by *divisible without remainder or fraction*, or as is often conveyed with a view to brevity by the less explicit designation *divisible*, the designation *exactly divisible* is often used; but these two words constitute a rather misleading or ambiguous expression, because, for instance, we can quite exactly divide 14 by 3, getting as the result—*not approximately*, but exactly—$4\frac{2}{3}$; or we can exactly divide a length of 14 inches by 3, getting as the exact result $4\frac{2}{3}$ inches.

To bring out clearly the full significance of the expression *equably and integrally divisible by a number*, we may notice that we could divide 14 into 3 equal parts consisting of 4 each, together with one additional part consisting of 2. The 14 would thus be divided integrally but not equably. The number of equal parts would be 3, and the total number of parts would be 4. 14 can be equably, but fractionally, divided by 3—that is, divided into 3 equal parts which will be fractional—the result being $4\frac{2}{3}$; but 14 is not *equably and integrally divisible* by 3.

The word number here in the phrase "any number of times exactly" is to be understood as being essentially limited to *number properly so called*, not as extending to include fractional numerics. Thus, although 10 is contained $2\frac{1}{2}$ times *exactly* in 25, yet 25 is not called a multiple of 10, nor is 10 called a measure of 25; but it is to be noticed that $2\frac{1}{2}$ is not properly *a number*, and $2\frac{1}{2}$ times is not properly *a number of times*. It is not essential that the measure and multiple be integers. One of them, or both, may be fractional. Thus, it is quite allowable to employ the words measure and multiple in the senses conveyed by the following extended wording of the definition:—

When one *numeric* is contained in another, any *number* of times exactly, the greater is called a MULTIPLE of the less, and the less is said to be a MEASURE of the greater, or to MEASURE it. Also any *numeric* is said to be a MULTIPLE of itself, and any *numeric* is said to be a MEASURE of itself.

Thus it is quite allowable to say that $7\frac{1}{2}$ is a multiple of $2\frac{1}{2}$, being *exactly* 3 times $2\frac{1}{2}$; and likewise that $2\frac{1}{2}$ is a measure of $7\frac{1}{2}$. For example, if the width between the parapets of a bridge is $7\frac{1}{2}$ yards, this can be quite exactly measured by a wand $2\frac{1}{2}$ yards long, not marked with any shorter length for use in measuring; and the length $7\frac{1}{2}$ yards, being exactly 3 times the length of the wand, is the multiple 3 of that length $2\frac{1}{2}$ yards. In the same way, a length of 11 feet can be exactly measured by a wand $5\frac{1}{2}$ feet long; and so $5\frac{1}{2}$ is a measure of 11, and 11 is a multiple of $5\frac{1}{2}$.

When a number is a multiple of two or more other numbers, it is called a COMMON MULTIPLE of those numbers; and a number which is a measure of two or more numbers is said to be a COMMON MEASURE of them.

Thus 12 is a common multiple of 2, 3, 4, and 6; and 3 is a common measure of 9, 15, and 24.

The number of single objects in any group of equal groups of the objects is called a COMPOSITE NUMBER;* or, in other words, any number which can be produced by the multiplication together of any two integers (that is, numbers properly so called), each greater than unity,

* This may be advantageously illustrated by using pebbles, or shells, or buttons, or the like, as single objects, and laying down groups of two for each group, or three for each; or any one same number for every group. Each group may have its single objects placed at random in it; or else, for some explanations, it will be good to place the objects of each group in a row, and to arrange the rows side by side, as is shown by dots in the illustration of integral multiplication on p. 23.

is called a COMPOSITE NUMBER; or, again in other words, any number which is divisible without remainder or fraction, by any integer other than either itself or unity, is called a COMPOSITE NUMBER. All other proper numbers are called PRIME NUMBERS; so we may say that PRIME NUMBERS are those whole numbers which cannot be divided into any equal integral parts each greater than unity.

Thus 4, 6, 8, and 9 are composite numbers; but 2, 3, 5, 7, and 11 are prime numbers. Also 1, or unity, if regarded as a number, falls to be classed with the primes, as it is clearly not a composite number. In some modes of thought and expression, however, unity, or the number 1, is not regarded as a number, and it is often left unmentioned in statements of all prime numbers up to any stated limit. In the table of prime numbers given below, the number 1, or unity, is inserted in brackets, so that it may be either included or omitted according to the mode of consideration adopted. It is of little practical importance, however, whether we call unity a prime number or not a number at all, except that, occasionally, difficulties or complications in expressing clearly what is intended arise through uncertainties as to the meanings of the words or expressions employed.

There is no brief and simple process by which to test any given number in general so as to find whether it be a prime or not, but there are systematised methods by which, through processes commencing with the smallest numbers and going forward by successive steps to greater numbers, all numbers below any specified limit can be distinguished into prime numbers and composite numbers, so as to make it quite certain for each of them which kind it is.* In such ways, by application of a moderate amount of arithmetical labour, tables of prime numbers have been calculated extending to very high limits—to limits, indeed, much beyond those of any ordinary practical wants. The following table, extending up to 1000, may be desirable as available for reference to the pupil or student in arithmetic:—

* Without the introduction of elaborate explanations at this stage, it may be useful or interesting to some of the more advanced students to receive the following suggestions and information:—

Many numbers can, by inspection, or by obvious tests, be readily found to be composite. Then as to any given number, not known to be composite and not known to be prime, if we have previously ascertained all primes below its square root, and if we find by trial that it is not divisible by any of them, we ascertain that it is a prime; but if it is divisible by any of them whatever, we know it is composite.

The simplest and easiest method available for making out a table of all primes below any specified limit is one which was invented or proposed about two centuries before the Christian era by Eratosthenes, and which he called his *sieve* for sifting out the numbers which are not primes and leaving those which are. It is easily understood and easily worked. It will be explained at a more advanced stage in the present treatise, and the place may be found by reference to the table of contents at the beginning of the volume.

TABLE OF ALL PRIME NUMBERS UNDER 1000.

[1?], 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97, 101, 103, 107, 109, 113, 127, 131, 137, 139, 149, 151, 157, 163, 167, 173, 179, 181, 191, 193, 197, 199, 211, 223, 227, 229, 233, 239, 241, 251, 257, 263, 269, 271, 277, 281, 283, 293, 307, 311, 313, 317, 331, 337, 347, 349, 353, 359, 367, 373, 379, 383, 389, 397, 401, 409, 419, 421, 431, 433, 439, 443, 449, 457, 461, 463, 467, 479, 487, 491, 499, 503, 509, 521, 523, 541, 547, 557, 563, 569, 571, 577, 587, 593, 599, 601, 607, 613, 617, 619, 631, 641, 643, 647, 653, 659, 661, 673, 677, 683, 691, 701, 709, 719, 727, 733, 739, 743, 751, 757, 761, 769, 773, 787, 797, 809, 811, 821, 823, 827, 829, 839, 853, 857, 859, 863, 877, 881, 883, 887, 907, 911, 919, 929, 937, 941, 947, 953, 967, 971, 977, 983, 991, 997.

PROPOSITION 1. Any multiple whatever of an integer is necessarily an integer.

The truth of this is quite obvious.

PROPOSITION 2. Any measure of a fractional numeric is necessarily a fractional numeric.

The truth of this is obvious, because the fractional numeric is a multiple of its own measure; and so that measure cannot be integral, because any multiple of an integer would necessarily be integral, and so it could not be the fractional numeric.

PROPOSITION 3. The greatest common measure of two or more integers must necessarily be an integer.

To prove this:—Let the integers be called *A*, *B*, *C*, &c. Now, if any fractional numeric will measure the integer *A*, the smallest multiple of that fractional numeric constituting an integer must measure the integer *A*. Also, if any fractional numeric will measure the integer *B*, the smallest multiple of that fractional numeric constituting an integer must measure the integer *B*. Exactly the like may be said respecting any fractional numeric which will measure the integer *C*; and so on for any fractional numeric which will measure any other integers. Hence, for any one fractional numeric which will measure each of the integers *A*, *B*, *C*, &c., it follows that the smallest multiple of that fractional common measure of them constituting an integer will measure each of these integers, or will be a common measure of them; and it is greater than the fractional common measure, and so the fractional common measure is not the greatest common measure.

The finding of the greatest common measure, and the finding of the least common multiple of two or more *numbers* (that is, integers or numbers proper), are processes often required as steps in REDUCTION OF FRACTIONS, and in FRACTIONAL ADDITION AND SUBTRACTION; but, in ordinary arithmetical work for useful purposes,

there is little, if any, practical requirement for dealing with either the greatest common measure, or the least common multiple, of two or more fractional numerics. Accordingly, when the greatest common measure, or the least common multiple, is spoken of, the understanding is usually implied that it is the greatest common measure, or the least common multiple of two or more *integral numerics* that is contemplated; and, in what follows in the present chapter, this usual understanding will be adopted; and so the rules and work will relate only to common measures and common multiples of numbers properly so called.

RULE I. *To find the greatest common measure of two given numbers:* (1.) Divide the greater number by the less. (2.) If there be a remainder, divide the less by it; and thus proceed, always dividing the last divisor by the last remainder, till nothing remains. The divisor which leaves no remainder is the greatest common measure.

Remark.—If, in the operation, any divisor be a prime number, and leave a remainder, it is unnecessary to proceed farther, as there is no common measure greater than unity.

Exam. 1. Required the greatest common measure of 247 and 323.

Here the first remainder is 76; and the less number 247 being divided by this, the remainder is 19. Dividing 76, the last divisor, by this, we find that there is no remainder. Hence, 19 is the common measure required; and it would be found by division to be contained 13 times in 247, and 17 times in 323, without remainder in either case.

247)323(1
247
76)247(3
228
19)76(4
76
..

With respect to the *reason of the rule*, since 19 is a measure of the last dividend 76, it must also be a measure of the preceding dividend 247, because $247 = 3 \times 76 + 19$; and 247 is one of the given numbers. Then, since 19 measures 76 and 247, it must measure their sum 323, which is the other given number; and a similar illustration is applicable in all cases.

19 is also the *greatest* common measure. For suppose, if possible, a greater number, as 20, to measure 247 and 323; then, it is evident, it must measure the first remainder 76, as that is their difference, and also it must measure 3 times 76 or 228; and since it is supposed also to measure 247, it must measure the second remainder 19, as that is the difference of 228 and 247; that is, if 20 measure 247 and 323, it must measure 19, a number less than itself, which is evidently impossible: therefore 20 does not measure both the given numbers; and it may be shown, in the same manner, that no other number greater than 19 can measure them;

therefore 19 is the *greatest* common measure. The remark which follows the rule may be proved in nearly a similar manner.*

RULE II. *To find the greatest common measure of three or more given numbers:* (1.) By the preceding rule find the g. c. meas.† of two of the numbers. (2.) Then by that rule again find the g. c. meas. of that result and another of them. (3.) If there are more given numbers, proceed in like manner till all of them have been so used. Then the last result so found will be the g. c. meas. of all the given numbers.

Exam. 2. Required the g. c. meas. of 81, 342, and 96.

```
1st Process.            2nd Process.
81)342(4                9)96(10
   324                    90
   18)81(4                6)9(1
      72                    6
      9)18(2                3)6(2
        18                    6
        ..                    .
```

Here by the first process we find 9 as the g. c. meas. of 81 and 342. Then by the 2nd process we find 3 as the g. c. meas. of 9 and 96. This result 3 is the g. c. meas. of all the given numbers.

---

* The following is an easy proof of the proposition stated in the remark annexed to Rule I. :—

Let us take, for scrutiny and explanation, any case in which a prime number comes out as a divisor and leaves a remainder. We may take, for instance, the case of its being required to find the g. c. meas. of 129 and 572. Carrying out the prescribed process, as in the margin, we get a remainder 17, which is a prime number, and which, on being used as a divisor, leaves a remainder 5. Then, according to the proposition comprised in the Remark, now to be proved, we assert that no number greater than 1 can be a common measure of the two original numbers.

```
129)572(4
    516
     56)129(2
        112
         17)56(3
            51
             5
```

To prove this :—Let us denote by the letter z any integral common measure whatever of the two original numbers; so that, if there are several integral common measures, z will have each of them separately as one of its meanings. Of course z must always have at least one value, namely, the number 1.

Now since z measures 572 and 129, it must measure 516, which is four times 129. So z measures 572 and 516, and therefore it measures their difference 56, which is the first remainder. By proceeding forward in like manner we may see that z measures necessarily every remainder in the whole process. But 17 is one of the remainders, and is a prime number; so z must measure that prime number 17, and, for this reason by itself, we see that no number except 1 or 17 can be a value of z, because these are the only two integral measures of the prime number 17. But, further, since a remainder 5 is left in the division by 17, we have it that z must measure 5; and as 17 cannot measure 5, a less number than itself, z cannot be 17. So there is no value remaining possible for z except the number 1; and so z is the number 1. So 1 is the only integral common measure. But by Prop. 3 of this chapter the greatest common measure must be an integer, and as 1 is the only integer which is a common measure, we see that 1 must be the greatest common measure of the two given numbers.

† For brevity g. c. meas. may advantageously be written instead of greatest common measure. Also l. c. mult. may be written for least common multiple.

To prove this* we have first to show that the result 3 is a common measure of the three given numbers, and afterwards to show that no greater number is a measure of them all. By the 2nd process we know that the result 3 is a measure of 96, the third of the three given numbers; and also that the same result 3 is a measure of 9, which by the 1st process we know is a measure of the 1st and 2nd given numbers, 81 and 342. So the 3 being a measure of a measure of each of these two must obviously be a measure of each of them. So we have now proved that the result 3 is a common measure of the three given numbers.

Next there is no greater common measure than 3 for all the given numbers. For suppose if possible a greater number, which we may call z, to measure the three given numbers. By proceeding in reference to Process 1 in like manner as in the proof of Rule I., we can see that if z is a measure of 81 and 342 it is also a measure of the result 9 of Process 1. Then again, by proceeding in like manner in reference to Process 2, we can see that if z is a measure of 9 and 96, it must also be a measure of the result 3, a number less than itself, which is impossible: therefore z does not measure all the three given numbers, and so no number greater than 3 can measure them. Hence 3 is their greatest common measure.

Exercises. Find the greatest common measures of the following numbers:—

1. 285 and 465 .... *Answ.* 15
2. 532 and 1274 .... *Answ.* 14
3. 888 and 2775 .... *Answ.* 111
4. 2145 and 3471 .... *Answ.* 39
5. 1879 and 2425 .... *Answ.* 1
6. 4872 and 81 .... *Answ.* 3
7. 234, 429 and 247 .... *Answ.* 13
8. 910, 1442 and 7245 .... *Answ.* 7
9. 1976, 2340 and 1742 .... *Answ.* 26
10. 966, 2940 and 2660 .... *Answ.* 14
11. 966, 2940, 2660 and 1309 .... *Answ.* 7
12. 112, 256 and 2386 .... *Answ.* 2
13. 403, 39 and 182 .... *Answ.* 13
14. 483, 63 and 672 .... *Answ.* 21
15. 250, 256 and 263 .... *Answ.* 1
16. 168, 2310, 114 and 294 .... *Answ.* 6

RULE III. *To find the least common multiple of two or more given numbers:* (1.) Arrange the given numbers in succession, and find by inspection a number which will measure as many of them as possible. (2.) By

* It may be sufficient for many learners, especially beginners in arithmetic, without their going through this demonstration, to be told merely that the proof is like that of Rule I., but somewhat longer.

this, divide all the given numbers which it measures, and write the quotients and the undivided numbers in a new line. (3.) Proceed, if possible, in the same manner with the numbers in this line; and thus continue the process, till no number, greater than unity, will measure any two or more of the numbers last found. (4.) Then multiply all the numbers in the last line, and all the divisors employed in the operation, continually together, and the result will be the common multiple required.

The work is often shortened by rejecting, in any line, any number that is contained without remainder in any other number in the same line.

If no two of the given numbers have any common measure greater than unity, the continual product of all the given numbers will be the least common multiple.

RULE IV. *To find the least common multiple, otherwise:* (1.) Find by Rule I., the greatest common measure of two of the given numbers. (2.) By this, divide one of those two numbers, and multiply the quotient by the other. (3.) Perform a similar operation on the product and another of the given numbers. (4.) Continue the process thus with each successive quotient, till all the numbers have been used, and the final result will be the least common multiple required.

Exam. 3. Required the least common multiple of 24, 10, 9, 32, 6, 45, and 25.

By Rule III. Here the given numbers being placed in the same line, we see by inspection, that 9 is contained exactly in 45, and 6 in 24: 6 and 9 are therefore neglected. Then, using 2 as a divisor, we obtain the quotients 12, 5, and 16, and we place in the line with them the undivided numbers, 45 and 25: and since the quotient 5 is contained exactly in either of these, it is rejected. We then divide by 3, and obtain the quotients 4 and 15, which with the undivided numbers, 16 and 25, form a new line. 4 is then rejected, as it is a measure of 16: and dividing by 5, we place in the next line the quotients 3 and 5, and the undivided number 16. Then, as no two of these have any common measure greater than unity, the three divisors, and the three numbers in the last line, are multiplied continually together, and the product 7200 is the common multiple required.

2)	24	10	(9)	32	(6)	45	25
3)	12	(5)		16		45	25
5)	(4)			16		15	25
				16		3	5

$2 \times 3 \times 5 \times 16 \times 3 \times 5 = 7200$, *answ.*

By Rule IV. Rejecting 9 and 6 as before, and commencing with 24 and 10 as a pair, we divide either 24 or 10 by 2, the greatest common measure of this pair; so dividing the 24 by 2 we get as quotient 12, and this quotient we multiply by 10, and the product 120 is the least common multiple of 24 and 10. Then. the greatest common measure of 120 and 32 being 8, we have $32 \div 8 = 4$, and $120 \times 4 = 480$, the least common multiple of 24, 10, and 32. In the next place, the greatest common measure of 480 and 45 is 15; and, since $45 \div 15 = 3$, we have $480 \times 3 = 1440$, the least common multiple of 24, 10, 32, and 45. Lastly, since the greatest common measure of 1440 and 25 is 5, and since $25 \div 5 = 5$, we have $1440 \times 5 = 7200$, as the common multiple required: and this result we see agrees with that of the previous process.

With respect to *the reasons of these rules*, it is difficult to give a strict, and, at the same time, an easily comprehended proof; and for most learners the following illustration will be preferable.

In the operation by Rule III., $2 \times 12 \times 5 \times 16 \times 45 \times 25$, the product of the first divisor and the numbers in the second line, is evidently a multiple of each of the given numbers, 24 being contained in it $5 \times 16 \times 45 \times 25$ times; 10, the second number, $12 \times 16 \times 45 \times 25$ times, &c. Again, $2 \times 3 \times 4 \times 16 \times 15 \times 25$, the product of the first two divisors and the numbers in the third line, is also a multiple of each of the given numbers, 24 being contained in it $16 \times 15 \times 25$ times; 10, the second number, $3 \times 4 \times 16 \times 3 \times 25$ times (since $2 \times 15 = 30$, and $30 = 10 \times 3$); 32, the fourth number, $3 \times 4 \times 15 \times 25$ times, &c.: and the illustration may be extended in a similar manner to the rest of the operation. That the given number 6 may be rejected in the operation, will appear from considering, that 6 is contained exactly 4 times in 24, and will therefore be contained without remainder in any multiple of 24, and 4 times as often as 24. In like manner it would appear, that 9 may be rejected, as 45 is a multiple of it. That 5 and 4 may be rejected in the succeeding lines will be evident from this, that they would disappear were the lines that contain them divided respectively by 5 and 4.

The proof of Rule IV. depends on the principle, that *if the product of any two numbers be divided by any factor which is common to both, the quotient will be a common multiple of the two numbers.* Thus, if 48, the product of 6 and 8, be divided by 2, a factor of both, the quotient 24 will be a multiple of each, since it may be regarded either as 8 multiplied by the quotient of 6 by the factor 2, or as 6 multiplied by the quotient of 8 by the same factor. Now, this being so, it is obvious, that the greater the common measure is, the less will be the multiple; and, consequently, the greatest common measure will produce the least common multiple. When the common multiple of the first two numbers is found, it is evident, that any number which is a common multiple of it and the third number, will be a multiple of the first, second, and third numbers; and thus the reason of the rule is manifest.

Remark on Rules III. and IV.—It may be remarked, that the second of these rules always gives the *least* common multiple. The multiple found by the first is not always the *least* possible; but it

will be such, if care be always taken to use such a divisor, as will measure *at least as many of the given numbers as any other divisor would.* This rule, therefore, being very easy in practice, is, in general, preferable to the other. It may be farther remarked, that by practice the pupil will become able to discover the common multiples by inspection, when the given numbers are not large.

Exercises. Required the least common multiples of the following numbers:—

Exercises.	*Answ.*	*Exercises.*	*Answ.*
17. 6 10 15 18	90	20. 63 12 84 7	252
18. 7 11 13 3	3003	21. 54 81 63 14	1134
19. 8 12 20 24 25	600	22. 2 3 4 5 6 7 8 9	2520

Exercises.	*Answ.*
23. 5 16 8 14 21	1680
24. 4301 1672 952	4,576,264
25. 1517 851 1219 19	35,135,237

REDUCTION OF FRACTIONS.

INTRODUCTORY REMARKS.—The name REDUCTION, as used in the title of the present chapter, does not bear exactly the same meaning as the same word does in the previous chapter entitled REDUCTION, at page 62. In that chapter the processes treated of, under the name *reduction*, have for their object the changing of the unit or units, or the changing of the denomination or denominations in which any quantity of any kind of thing is expressed, and the finding of a true numerical expression for the same quantity in the altered unit or units. That might be described as *reduction between different denominations, the quantity expressed being unchanged;* or briefly might be called DENOMINATIONAL REDUCTION. In contrast with that, the *reductions* or *changes* in fractions, or in any fractional numerics to be treated of in the present chapter, are to consist only of changes in form in the numerical expression, while the numerical value is to remain unchanged. Or, briefly, the processes will consist in changing fractionally the form of numerics, usually themselves fractional at first, without change of numerical value. Thus if we find that $\frac{9}{12}$ may, without change of value, be otherwise expressed by varied fractions, such as $\frac{6}{8}$ or $\frac{3}{4}$, we are performing operations in the kind of Reduction that is included in the present chapter. Such operations might well be called NUMERICAL REDUCTION.

A fraction is said to be in its LOWEST TERMS, or in its SIMPLEST FORM, when it is expressed in the least whole numbers* possible.

* We might here better say *whole numerics*, or *integers*, but the old nomenclature, in the expression *whole numbers*, is here retained as being for the present the more usual.

RULE I. *To reduce a fraction to its lowest terms:* (1.) Divide the terms of the given fraction by any number that will measure both: the quotients will be the numerator and denominator of an equivalent fraction in *lower* terms. (2.) Let this fraction, if possible, be reduced in like manner: and proceed thus, till a fraction is obtained for whose terms no common measure can be found.

The application of this rule will often be facilitated by the following directions and remarks:—(1.) If the terms of the fraction end in ciphers, cut an equal number from each. (2.) If they end each in 5, or one in 5, and the other in a cipher, divide them both by 5; or double them, and reject a cipher from each of the results. (3.) If 2 measure the last figure of each term, it will measure the terms themselves. In like manner, if 4 measure the number expressed by the last two figures, or 8 that expressed by the last three, 4 in the one case, and 8 in the other, will measure both terms. (4.) If 3 or 9 measure the sum of the digits of each term, 3 in the one case, and 9 in the other, will measure both terms.—(*See foot note, pages 28 and 29.*)

Exam. 1. Reduce $\frac{108}{162}$ to its lowest terms.

Here, we divide the given terms by 2; those of the result by 9; and those of that result by 3; and we see that $\frac{108}{162}$ is equal to $\frac{54}{81}$, $\frac{6}{9}$, or $\frac{2}{3}$. The terms of the last of these have evidently no common measure greater than unity; it is therefore the simplest form of the fraction. The same would be obtained rather more quickly by dividing by 6 and 9.

$$2)\frac{108}{162}=9)\frac{54}{81}=3)\frac{6}{9}=\frac{2}{3}.$$

Exam. 2. Reduce $\frac{21000}{37800}$ to its simplest form.

Here, two ciphers are cut from the end of each term, which is equivalent to the dividing of each by 100. The quotients are then divided by 6, and the results by 7: and the fraction is reduced to the form $\frac{5}{9}$, which is its simplest form.

$$6)\frac{210,00}{378,00}=7)\frac{35}{63}=\frac{5}{9}.$$

The *reason* of this rule is evident from Proposition 2, page 50.

Exercises. Reduce the following fractions to their simplest forms:

Exer.	*Answ.*	*Exer.*	*Answ.*	*Exer.*	*Answ.*
1. $\frac{63}{105}$	$\frac{3}{5}$	4. $\frac{240}{312}$	$\frac{10}{13}$	7. $\frac{176}{1000}$	$\frac{22}{125}$
2. $\frac{55}{121}$	$\frac{5}{11}$	5. $\frac{126}{162}$	$\frac{7}{9}$	8. $\frac{960000}{1152000}$	$\frac{5}{6}$
3. $\frac{120}{320}$	$\frac{3}{8}$	6. $\frac{57}{63}$	$\frac{19}{21}$	9. $\frac{144000}{1920000}$	$\frac{3}{40}$

The foregoing method taught under Rule I. is very convenient and easy in practice, when the terms of the proposed fractions are not very large, or when the divisors are readily discovered. It fails, however, in many cases, to determine whether the fraction is in its lowest terms or not; and, while it often depends on guesses, and leaves uncertainty, there is usually considerable difficulty ex-

perienced in the finding of some of the measures so to be obtained, except in the cases of some of the smaller numbers. Thus, if the fraction $\frac{399}{627}$ were proposed, we should readily discover, that it may be reduced, by division by 3, to the form $\frac{133}{209}$, which we would perhaps conceive to be its simplest form, not knowing that it is still further reducible by division by 19, and that the simplest form is $\frac{7}{11}$. The following method, though often tedious, is perfect in principle, always reducing the fraction to its simplest form, and not depending on guesses with trials.

RULE II. *To reduce a fraction to its lowest terms, otherwise:* (1.) Find the greatest common measure of the numerator and denominator. (2.) Divide them both by this, and the quotients will be the numerator and denominator of the required fraction.

It is often of advantage to carry the reduction as far, by Rule I., as can readily be done, and then to apply Rule II. to the result.

Exam. 3. Reduce the fraction $\frac{679}{1746}$ to its lowest terms.

In this example we find the greatest common measure to be 97; and dividing both terms of the given fraction by this, we obtain $\frac{7}{18}$, which is the equivalent fraction in its lowest terms.

Exam. 4. Reduce $\frac{8280}{11385}$ to its lowest terms.

Here, because the terms end in 5 and 0, we notice that 5 is a measure, and then division by 5 reduces the fraction to $\frac{1656}{2277}$. In this again, both terms are divisible by 9, as may be seen by noticing that 9 measures the sum of the digits of each term; and the result is $\frac{184}{253}$, which, by Rule II., is reduced to $\frac{8}{11}$.

Exam. 5. Reduce $\frac{468}{1266}$ to its most simple expression.

We see here, immediately, that 6 is a common measure; for 3 measures the sum of the digits of each term, and 2 must be a measure, since the unit figures are even. After division by 6, therefore, the fraction becomes $\frac{78}{211}$. In applying Rule II. to this, we get, for the third division, 55 to be divided by 23, which is a prime number, and is not a measure of 55. We conclude, therefore, without proceeding farther, that $\frac{78}{211}$ is the simplest form of the fraction.

It ought to be borne in mind, that usually at the conclusion of an operation, if there be a fraction, it should be reduced to its simplest form.

Exercises. Reduce the following fractions to their lowest terms:—

Exer.	Answ.	Exer.	Answ.	Exer.	Answ.
10. $\frac{272}{425}$	$\frac{16}{25}$	13. $\frac{6465}{7335}$	$\frac{481}{489}$	16. $\frac{10395}{16819}$	$\frac{945}{1529}$
11. $\frac{594}{2142}$	$\frac{33}{119}$	14. $\frac{1236}{1632}$	$\frac{103}{136}$	17. $\frac{5415}{30105}$	$\frac{361}{2007}$
12. $\frac{5170}{8734}$	$\frac{235}{397}$	15. $\frac{285714}{999999}$	$\frac{2}{7}$	18. $\frac{15863}{21489}$	$\frac{547}{741}$

RULE III. *Any number of fractions having different denominators being given: to find equivalent fractions*

having a common denominator the smallest possible as an integral numeric: (1.) Find (as is taught in the chapter on Measures and Multiples, pages 134 and 135) the least common multiple of all the denominators: this will be the common denominator. (2.) Then, divide the common multiple by the first of the given denominators, and multiply the quotient by the first of the given numerators: the product will be the first of the required numerators. The other numerators will be found in a similar manner.

RULE IV. *Any number of fractions being given: to find equivalent fractions having a common denominator, not necessarily the smallest possible as an integral numeric:* (1.) Multiply each numerator by all the denominators, except its own, and the product will be the numerator of the equivalent fraction. (2.) Multiply all the denominators continually together for the common denominator.

Exam. 6. Reduce $\frac{5}{8}$, $\frac{7}{12}$, $\frac{11}{18}$, and $\frac{9}{20}$, to fractions having a common denominator.

By Rule III. of the present chapter. Here, by what is taught in Rule III. or IV. of MEASURES AND MULTIPLES, pages 134 and 135, the least common multiple is found to be 360; and the rest of the work, together with the resulting fractions, will stand as in the margin. The correctness of the results would be proved by reducing them to their simplest forms, as the given fractions would thus be got back again. Thus $\frac{225}{360}$ would be reduced to $\frac{5}{8}$, $\frac{210}{360}$ to $\frac{7}{12}$, &c.

8)360	12)360	18)360	20)360	
45	30	20	18	
5	7	11	9	
225	210	220	162	
$\frac{225}{360}$,	$\frac{210}{360}$,	$\frac{220}{360}$,	$\frac{162}{360}$,	*answ.*

By Rule IV. $5 \times 12 \times 18 \times 20 = 21600$, the first numerator; $7 \times 8 \times 18 \times 20 = 20160$, the second; $11 \times 8 \times 12 \times 20 = 21120$, the third; $9 \times 8 \times 12 \times 18 = 15552$, the fourth; and $8 \times 12 \times 18 \times 20 = 34560$, the common denominator. Hence, the fractions found by this rule are $\frac{21600}{34560}$, $\frac{20160}{34560}$, $\frac{21120}{34560}$, and $\frac{15552}{34560}$; which being reduced to their lowest terms, would, in like manner, be shown to be equivalent to the given fractions.

In this example, the results found by Rule IV. come out expressed in numbers inconveniently large; and the same is commonly the case. Hence Rule III. should always be preferred, except when it is noticed that no two of the given denominators have a common measure.

With respect to *the reasons of these last two rules*, it is evident that the operation by the second of them (Rule IV.) is just the multiplication of the terms of each fraction by all the denominators

except its own. Thus, the first of the given fractions is $\frac{5}{8}$, which, by the operation, is changed into $\dfrac{5 \times 12 \times 18 \times 20}{8 \times 12 \times 18 \times 20}$, or $\dfrac{21600}{34560}$, and this, by Proposition 1, pages 49 and 50, is equal to the proposed fraction, $\frac{5}{8}$.

In Rule III. the division of the least common multiple by the given denominator is evidently just the finding of the number, by which, if both the numerator and the denominator of the given fraction be multiplied, the denominator of the result will be the least common multiple; and, by Proposition 1, pages 49 and 50, the result will be equal to the given fraction. Thus, in the preceding example, when 360 is divided by 8, the quotient is 45; by which, if both terms of the fraction $\frac{5}{8}$ be multiplied, there results $\frac{225}{360}$ for the equivalent fraction.

Exercises. Reduce the following sets of fractions to others, having common denominators :—

	Exercises.	*Answers.*
19.	$\frac{5}{12}, \frac{9}{16}, \frac{11}{24}, \frac{17}{33}, \frac{3}{8}$	$\frac{220}{528}, \frac{297}{528}, \frac{242}{528}, \frac{272}{528}, \frac{198}{528}$
20.	$\frac{11}{27}, \frac{19}{24}, \frac{5}{6}, \frac{7}{15}, \frac{3}{5}, \frac{1}{9}$	$\frac{440}{1080}, \frac{855}{1080}, \frac{900}{1080}, \frac{504}{1080}, \frac{648}{1080}, \frac{120}{1080}$
21.	$\frac{17}{80}, \frac{19}{60}, \frac{11}{60}, \frac{53}{75}, \frac{4}{9}, \frac{5}{36}$	$\frac{765}{3600}, \frac{1140}{3600}, \frac{660}{3600}, \frac{2544}{3600}, \frac{1600}{3600}, \frac{500}{3600}$
22.	$\frac{11}{35}, \frac{19}{45}, \frac{25}{42}, \frac{1}{100}, \frac{79}{81}$	$\frac{17820}{56700}, \frac{23940}{56700}, \frac{33750}{56700}, \frac{567}{56700}, \frac{55300}{56700}$
23.	$\frac{1}{5}, \frac{1}{7}, \frac{1}{9}, \frac{1}{11}$	$\frac{693}{3465}, \frac{495}{3465}, \frac{385}{3465}, \frac{315}{3465}$
24.	$\frac{11}{12}, \frac{17}{18}, \frac{29}{30}, \frac{47}{48}, \frac{7}{16}$	$\frac{660}{720}, \frac{680}{720}, \frac{696}{720}, \frac{705}{720}, \frac{315}{720}$
25.	$\frac{17}{100}, \frac{17}{1000}, \frac{17}{10000}$	$\frac{1700}{10000}, \frac{170}{10000}, \frac{17}{10000}$
26.	$\frac{41}{60}, \frac{13}{90}, \frac{13}{25}, \frac{1}{105}, \frac{4}{9}$	$\frac{4305}{6300}, \frac{910}{6300}, \frac{3276}{6300}, \frac{60}{6300}, \frac{2800}{6300}$
27.	$\frac{4}{7}, \frac{1}{10}, \frac{13}{15}, \frac{5}{16}, \frac{9}{28}, \frac{2}{35}$	$\frac{960}{1680}, \frac{168}{1680}, \frac{1456}{1680}, \frac{525}{1680}, \frac{540}{1680}, \frac{96}{1680}$
28.	$\frac{1}{3}, \frac{2}{9}, \frac{4}{27}, \frac{8}{81}, \frac{16}{243}, \frac{32}{729}$	$\frac{243}{729}, \frac{162}{729}, \frac{108}{729}, \frac{72}{729}, \frac{48}{729}, \frac{32}{729}$

RULE V. *To reduce an improper fraction to a whole or mixed number:* Divide the numerator by the denominator; the quotient will be the whole number required: and if there be any remainder, write it over the given denominator for the fractional part of the required result.

The *reason of this rule* and that of the next are evident from the nature of fractions.

Exam. 7. Reduce $\frac{56}{7}$ and $\frac{200}{9}$ to whole or mixed numbers.

$$\frac{56}{7} \qquad 7)56 \qquad 8, \textit{answ.}$$

$$\frac{200}{9} \qquad 9)200 \qquad 22\tfrac{2}{9}, \textit{answ.}$$

Exercises. Reduce the following fractions to whole or mixed numbers :—

Exer.	*Answ.*	*Exer.*	*Answ.*	*Exer.*	*Answ.*
29. $\frac{19}{19}$	1	31. $\frac{7536}{375}$	$20\frac{12}{125}$	33. $\frac{509}{12}$	$42\frac{5}{12}$
30. $\frac{6437}{298}$	$21\frac{179}{298}$	32. $\frac{10000}{111}$	$90\frac{10}{111}$	34. $\frac{750}{25}$	30

RULE VI. *To reduce a mixed number to an improper fraction:* Multiply the whole number by the denominator of the fractional part, and to the product add the numerator: the sum will be the required numerator; below which write the given denominator.

A whole number may be expressed in a fractional form by writing the figure 1 below it as a denominator; or by multiplying it by any whole number, and writing that number below the product, as denominator. Thus, $9 = \frac{9}{1} = \frac{36}{4}$, &c.

Exam. 8. Reduce $5\frac{11}{12}$ to an improper fraction.

$$\begin{array}{r} 5\frac{11}{12} \\ 12 \\ \hline \frac{71}{12}, \textit{ answ.} \end{array}$$

Exercises. Reduce the following numbers to improper fractions:—

Exer.	*Answ.*	*Exer.*	*Answ.*	*Exer.*	*Answ.*
35. $15\frac{1}{6}$	$\frac{91}{6}$	37. $46\frac{5}{8}$	$\frac{373}{8}$	39. $1\frac{999}{1000}$	$\frac{1999}{1000}$
36. $3\frac{17}{55}$	$\frac{182}{55}$	38. $21\frac{117}{583}$	$\frac{12360}{583}$	40. $19\frac{5}{7}$	$\frac{138}{7}$

FRACTIONAL ADDITION.

The name FRACTIONAL ADDITION is applicable to the addition of any numerics whatever, except such as are all integers in reality, and are besides clear of fractional expression.* In other words, this name is applicable to the addition of numerics which include among them any fractional expression or expressions.

The main principle on which fractional addition depends may be readily understood through noticing a few obvious examples.

We may readily see that 3 eighths plus 2 eighths make 5 eighths, just as 3 pebbles plus 2 pebbles make 5 pebbles. That statement about the eighths may, in ordinary fractional form, be written thus: $\frac{3}{8} + \frac{2}{8}$ make $\frac{5}{8}$. Thus we see that different fractions,

* It is to be observed that we often have to deal with numerics which, though really integers, are fractionally expressed; and operations in Addition, when some of the given numerics are fractionally expressed, may sometimes conveniently be regarded as cases of Fractional Addition. Thus the numerics $3\frac{8}{4}$ and $4\frac{3}{1\frac{1}{2}}$, though fractionally expressed, are really integers. The former of these, $3\frac{8}{4}$, is obviously equal to the whole number, 5. The latter, $4\frac{3}{1\frac{1}{2}}$, is a complex fractional numeric, which, through the teaching given a little further on in this treatise, may easily be shown to be equal to the whole number 6.

all having one same denominator, are to be added together by adding their numerators to get the numerator for a fraction which will be the required sum, and putting the common denominator as its denominator. Also we may readily see that if the fractions given to be added together have not a common denominator, they may be first reduced in expression to fractions equal to them and having a common denominator, and then may be added together in the way already explained. Further, it may be noticed that if it were required to add together 6, and $4\frac{3}{8}$, and $7\frac{2}{8}$, we might first add together the fractional parts, $\frac{3}{8}$ and $\frac{2}{8}$; and, having found their sum to be $\frac{5}{8}$, we might add to that the sum of the integral parts, 6, and 4, and 7, which is 17; and, adding this along with the sum of the fractional parts, we would get $17\frac{5}{8}$ for the sum total. So we may see that if the given numerics comprise any integers, the fractional parts may be added together, and the integral parts may be added together, and then the two sums may be added together to get the sum total of the given numerics.

If, on adding together the fractional parts in the way already explained, the sum comes out to have its numerator greater than its denominator, this sum may be reduced to an integer with a proper fraction; and if there be, besides, any additional integer in the result, the integers may be added together so as to obtain, for the final result, one integer with one proper fraction.

The methods of procedure thus suggested may be stated in the form of a rule as follows:—

Rule. (1.) Reduce all fractions in the given numerics to a set of fractions having a common denominator, if they be not such already. (2.) Then add the numerators together, and put their sum as the numerator for a fraction, and put the common denominator as its denominator. The fraction so obtained will be the sum of the fractions in the given numerics; and it will be the whole required sum unless there be any integral number or numbers included in the given numerics. (3.) If this fraction so obtained be greater than unity (that is, if it be an improper fraction) reduce it to a whole or a mixed number, and that will be the required sum, unless there be any integer included in the given numerics. (4.) If there be any such integer or integers, add such as there may be and the integral portion of the previously obtained fraction, if any, all together; and annex to the total integer so obtained the proper fraction already obtained in the summation of the given fractions. The result will be the sum total required.

Exam. 1. Add together $\frac{14}{15}$, $\frac{11}{15}$, and $\frac{8}{15}$.

Here it is to be noticed that the given numerics are all proper fractions, and have a common denominator already. Adding, then, the numerators together, we get 33; and taking this as the numerator for a fraction, and taking for its denominator the common denominator of the given fractions, which is 15, we obtain $\frac{33}{15}$ as the sum of the given numerics. But, as the numerator in this is greater than the denominator (or, as the expression is an improper fraction), we reduce it to a mixed number by actually dividing the numerator by the denominator. So we find for the sum required $2\frac{3}{15}$, or $2\frac{1}{5}$.

$$\left.\begin{array}{r}14\\11\\8\end{array}\right\}15$$
$$\begin{array}{l}\overline{33}\\ \overline{15)33(2\frac{3}{15},\text{ or }2\frac{1}{5},}\\ \qquad\qquad\textit{answ.}\end{array}$$

Exam. 2. Add together $3\frac{31}{40}$, $\frac{29}{36}$, and $6\frac{23}{25}$.

In this exercise, by reducing the given fractions to equivalent ones having a common demoninator, we find the first to be $\frac{1395}{1800}$, the second $\frac{1450}{1800}$, &c. Then, by adding the numerators, we find the sum of the fractions to be $\frac{4501}{1800}$, or $2\frac{901}{1800}$, by reduction to a mixed number. We then for convenience write the fractional part beneath the given fractions, as originally arranged in a column; and carrying 2 to the whole numbers, we find the required sum to be $11\frac{901}{1800}$.

$$\begin{array}{lcl}3\frac{31}{40} & \ldots\ldots & 1395\\ \frac{29}{36} & \ldots\ldots & 1450\\ 6\frac{23}{25} & \ldots\ldots & 1656\\ \hline 11\frac{901}{1800} & & \overline{4501}\\ \textit{Answ.} & & \overline{1800)4501(2\frac{901}{1800}}\end{array}\quad\left.\begin{array}{l}\\ \\ \\ \end{array}\right\}1800$$

Exercises.	*Answers.*
1. $\frac{8}{15}+\frac{9}{20}$	$=\frac{59}{60}$
2. $\frac{7}{8}+\frac{1}{12}+\frac{17}{18}+\frac{23}{24}+\frac{20}{27}$	$=3\frac{65}{108}$
3. $\frac{7}{8}+\frac{7}{12}+\frac{13}{16}+\frac{11}{18}+\frac{19}{24}$	$=3\frac{97}{144}$
4. $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\frac{1}{64}+\frac{1}{128}$	$=\frac{127}{128}$
5. $\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\frac{31}{32}+\frac{63}{64}+\frac{127}{128}$	$=6\frac{1}{128}$
6. $\frac{2}{3}+\frac{3}{4}+\frac{4}{5}+\frac{5}{6}+\frac{6}{7}$	$=3\frac{127}{140}$
7. $\frac{3}{8}+3\frac{14}{15}+10\frac{2}{5}+\frac{9}{22}$	$=15\frac{31}{264}$
8. $3\frac{5}{8}+12\frac{1}{6}+13\frac{83}{84}$	$=29\frac{131}{168}$
9. $2\frac{3}{5}+4\frac{7}{8}+5\frac{3}{10}$	$=12\frac{31}{40}$
10. $\frac{1}{3}+\frac{4}{7}+\frac{21}{32}$	$=1\frac{377}{672}$
11. $\frac{5}{8}+\frac{5}{14}+\frac{1}{6}$	$=1\frac{25}{168}$
12. $\frac{2}{5}+\frac{23}{36}+\frac{3}{5}$	$=1\frac{23}{36}$
13. $\frac{1}{4}+\frac{65}{189}$	$=\frac{449}{756}$
14. $\frac{3}{8}+\frac{1}{22}+\frac{7}{24}+\frac{29}{88}$	$=1\frac{1}{24}$
15. $4\frac{2}{15}+2\frac{7}{20}+1\frac{1}{12}+3\frac{5}{18}$	$=10\frac{38}{45}$
16. $\frac{3}{4}+1\frac{1}{3}+\frac{4}{7}+1\frac{3}{4}$	$=4\frac{17}{42}$
17. $\frac{23}{25}+\frac{2}{5}+\frac{7}{24}+\frac{11}{15}$	$=2\frac{69}{200}$

FRACTIONAL SUBTRACTION.

The name FRACTIONAL SUBTRACTION is applicable to the subtraction of any numeric from any other numeric, except when both are integers in reality, and are besides clear of fractional expression.

The explanations and illustrations already given in reference to Fractional Addition may afford aid towards the easy understanding of the instructions which will now be given for Fractional Subtraction.

In subtraction in general the number, group of objects, quantity, or numeric, from which a subtraction is to be made, is called the MINUEND; and the number, group of objects, quantity, or numeric, which is to be subtracted, is called the SUBTRAHEND. These names will be used in the following explanations and instructions on Fractional Subtraction, because without them or some equivalent pair of names the desirable brevity with clearness and freedom from ambiguity of meaning would be unattainable.*

For subtraction as an arithmetical process† the subtrahend must necessarily be not greater than the minuend. Several varieties of cases may be separately considered, the chief of which are indicated in the following rules, and these rules may also render obvious enough the principles by which to treat other varied cases.

RULE I. When the minuend is a mixed number consisting of a whole number and a proper fraction, and the subtrahend is a whole number; subtract the integral subtrahend from the integral portion of the minuend, and to the remainder so obtained annex the fractional portion of the minuend.

Remark on Rule I.—It is to be noticed that the operation directed in this rule will always be possible, because the subtrahend must be not greater than the minuend; and therefore obviously in

* Although the names *minuend* and *subtrahend* have often been regarded with disfavour, and treated as nearly obsolete, yet they are now very commonly in use; and without them, or some pair of equivalent words, the difficulty of composing clear explanations and instructions in arithmetic would be much greater than it is when their aid is accepted.

† Under the algebraic method of expression the sense of the word subtraction is extended from that in which it is used in ordinary arithmetic. In the algebraic method we say that if we subtract 7 from 5 we get minus 2 as the result. That may be interpreted or illustrated by saying that if from a certain marked point on a road we go, in one direction called forward, 5 miles, and then go backward 7 miles, we have in the end really gone backwards 2 miles from the starting point. But this result is arrived at by the purely arithmetical process of subtracting 5 from 7, the smaller from the greater; and then observing that as the backward distance travelled, 7 miles, is more than the forward distance, 5 miles, the resultant distance travelled is a distance of 2 miles backward from the starting point, and this in algebra is called a distance of minus 2 miles forward.

the case contemplated in the rule the integral subtrahend must be not greater than the integral portion of the minuend.

RULE II. When the subtrahend is a fraction less than unity (a proper fraction) and the minuend is a whole number—(1.) Consider unity to be taken from the given minuend and treated as a separate part of the minuend, and consider it as expressed for temporary use as a fraction having numerator and denominator each equal to the denominator in the subtrahend. (2.) From the numerator of this temporary fractional expression of unity subtract the numerator in the subtrahend, and make a fraction having the remainder for numerator, and having for denominator the previously used common denominator. (3.) Then annex the fraction thus made to what remains on the subtraction of unity from the given integral minuend. The numeric so obtained will be the result required as the remainder left by the entire process of subtraction.

RULE III. When the minuend and subtrahend each consist of a whole number and a proper fraction—(1.) Reduce the fractions to equivalent fractions having a common denominator, if they have not a common denominator already. (2.) Then, if possible, subtract the numerator in the subtrahend from the numerator in the minuend; and set the remainder as the numerator for a fraction, and put the common denominator for its denominator. Subtract the whole number in the subtrahend from the whole number in the minuend, and place the remainder as a whole number in front of the fraction just before found. The numeric so obtained will be the result required. (3.) But if the numerator in the subtrahend exceed the numerator in the minuend, consider unity to be taken from the given whole number in the minuend and to be treated as a separate part of the minuend, and consider it as expressed for temporary use as a fraction having numerator and denominator each equal to the common denominator already found; from the numerator of this temporary fractional expression of unity subtract the numerator in the subtrahend, and to the remainder add the given numerator in the minuend; set the sum as the numerator for a fraction, and put the common denominator for its denominator;

carry one to the whole number in the subtrahend, and subtract that number so augmented from the given whole number in the minuend; and place the remainder in front of the fraction just before found. The numeric so obtained will be the result required.

Exam. 1. From $105\frac{3}{7}$ take 24.

Here the nature of the requisite operation is obvious, and the meaning of the work in the margin is also obvious. This case falls within the scope of Rule I.

$$\begin{array}{r} 105\frac{3}{7} \\ 24 \\ \hline 81\frac{3}{7}, \textit{answ.} \end{array}$$

Exam. 2. From 37 subtract $\frac{14}{19}$.

Here the work is so easy that it can readily be done mentally, and the result can be noted without any written calculation. Thus we take unity from the 37, and treat it as a separate part of the minuend, and expressing it as $\frac{19}{19}$ we subtract $\frac{14}{19}$ from it, and we get $\frac{5}{19}$. Then in front of this we write the original whole number diminished by 1, that is, 36; and so we have for answer $36\frac{5}{19}$. This case obviously comes under Rule II.

$$36\frac{5}{19}, \textit{answ.}$$

Exam. 3. From $73\frac{5}{6}$ take $19\frac{4}{15}$.

This case evidently comes under Rule III. The given fractions $\frac{5}{6}$ and $\frac{4}{15}$, on being reduced to equivalent fractions having a common denominator, become $\frac{25}{30}$ and $\frac{8}{30}$. Then the numerator 8 in the subtrahend is taken from the numerator 25 in the minuend, and the remainder 17 so found is set as numerator for a fraction having for denominator the common denominator 30. Thus is obtained the fraction $\frac{17}{30}$. Then the integer 19 in the subtrahend is taken from the integer 73 in the minuend, and the remainder 54 is placed in front of the fraction $\frac{17}{30}$ before found. So the result of the entire subtraction required to be done is found to be $54\frac{17}{30}$.

$$\begin{array}{rcr} 73\frac{5}{6} & \ldots & 73\frac{25}{30} \\ 19\frac{4}{15} & \ldots & 19\frac{8}{30} \\ \cline{3-3} & & 54\frac{17}{30}, \textit{answ.} \end{array}$$

For brevity, when the principles are clearly understood, the work in cases such as this may be arranged as here shown in the margin, with less writing than in the fuller exposition above.

$$\begin{array}{rcr} 73\frac{5}{6} & \ldots & 25 \\ 19\frac{4}{15} & \ldots & 8 \\ \hline 54\frac{17}{30} & & 17 \end{array} \left.\vphantom{\begin{array}{c}1\\1\end{array}}\right\} 30$$

Exam. 4. From $10\frac{2}{25}$ subtract $3\frac{22}{35}$.

This case obviously comes under Rule III. The fractions $\frac{2}{25}$ and $\frac{22}{35}$ are reduced to $\frac{14}{175}$ and $\frac{110}{175}$. Now we cannot subtract $\frac{110}{175}$ from $\frac{14}{175}$, so we consider unity to be taken from the 10 in the minuend, leaving 9 instead, and that unity or one we treat as a separate part of the minuend, and expressing it mentally as $\frac{175}{175}$, we subtract the numerator 110 from the numerator 175

$$\begin{array}{rcr} 10\frac{2}{25} & \ldots & 14 \\ 3\frac{22}{35} & \ldots & 110 \\ \hline 6\frac{79}{175} & & 79 \end{array} \left.\vphantom{\begin{array}{c}1\\1\end{array}}\right\} 175$$

and get 65, to which we add the numerator 14, and so we get 79 as numerator for a fraction which is to have the common denominator 175 for its denominator. So we get the fraction $\frac{79}{175}$, which is to be the fractional part of the required result. Then to adjust for the abatement by unity of the 10 in the minuend, instead of subtracting the 3 in the subtrahend from 9, we proceed usually a little more easily by the method of carrying 1 to the 3 and getting 4, and subtracting the 4 from the 10. which will give the same result 6, and this 6 we write in front of the previously obtained fraction, and so we get for the final result $6\frac{79}{175}$.

Further Explanation.—For purposes of explanation the process may be written out more fully as here in the margin, several steps supposed in the previous arrangement of the work to have been effected mentally being here exhibited in full. It will be noticed that in the subtraction of the one integral portion from the other it is a matter of indifference whether we take 3 from 9 and get 6, or carry 1 to 3 and so get 4, and take this 4 from 10 and get 6, as the result 6 is the same in both ways, and both ways are true in principle.

$$\begin{array}{lllll} 10\frac{2}{25} & \ldots\ldots & 10\frac{14}{175} & \ldots\ldots & 9+\frac{175}{175}+\frac{14}{175} \\ 3\frac{22}{35} & \ldots\ldots & 3\frac{110}{175} & \ldots\ldots & 3+\frac{110}{175} \\ \hline & & & & 6+\frac{65}{175}+\frac{14}{175} \\ & & & \text{Or} & 6\frac{79}{175}, \textit{ answ.} \end{array}$$

Remark.—Cases may occur in practice slightly different from any of those formally dealt with in the foregoing rules. The treatment of such cases will, however, usually present no difficulty after such as are formally included under the rules are well understood; and, besides, a more advanced knowledge of the management of fractions, especially of complex fractions, may obviate difficulties in some cases both in fractional addition and fractional subtraction. One of the further cases, at the present stage easily manageable, is that in which an improper fraction occurs in one or both of the minuend and subtrahend. This case can be treated sometimes most easily by reducing the improper fraction to a mixed number, and sometimes most easily by reducing the fractions, one or both improper, to equivalent fractions having a common denominator, and then proceeding by operations which may be quite obvious after knowledge has been attained of the rules and explanations already given.

Exercises.	*Answers.*
1. From $23\frac{13}{17}$ subtract 5	$18\frac{13}{17}$
2. From 376 take $\frac{41}{164}$	$375\frac{3}{4}$
3. Subtract $\frac{62}{13}$ from 25	$20\frac{3}{13}$
4. From 89 subtract $\frac{49}{164}$	$88\frac{115}{164}$
5. From $\frac{37}{42}$ subtract $\frac{13}{126}$	$\frac{7}{9}$
6. From $3\frac{5}{14}$ subtract $1\frac{13}{21}$	$1\frac{31}{42}$
7. From $12\frac{3}{4}$ subtract $10\frac{13}{16}$	$1\frac{15}{16}$
8. From $\frac{1}{7}$ subtract $\frac{1}{11}$	$\frac{4}{77}$
9. From $12\frac{23}{28}$ subtract $9\frac{27}{35}$	$3\frac{1}{20}$
10. From $4\frac{1}{24}$ subtract $3\frac{1}{16}$	$\frac{47}{48}$
11. From $\frac{16}{21}$ subtract $\frac{8}{15}$	$\frac{8}{35}$
12. From $\frac{9}{55}$ subtract $\frac{1}{15}$	$\frac{16}{165}$
13. $\frac{19}{96} \sim \frac{43}{216}$*	$\frac{1}{864}$
14. $\frac{11}{75} \sim \frac{7}{45}$	$\frac{2}{225}$
15. From $5\frac{23}{32}$ subtract $2\frac{2}{7}$	$3\frac{97}{224}$
16. From $11\frac{2}{3} + 8\frac{7}{9}$ subtract $9\frac{19}{22}$	$10\frac{115}{198}$
17. From $\frac{95}{26}$ subtract $\frac{34}{13}$	$1\frac{1}{26}$
18. From $\frac{129}{14}$ subtract $\frac{52}{13}$	$5\frac{3}{14}$
19. From $\frac{115}{26}$ subtract $\frac{52}{13}$	$\frac{11}{26}$
20. Take away $\frac{51}{17}$ from $5\frac{3}{4}$	$2\frac{3}{4}$
21. Subtract $\frac{1}{2}$ from $\frac{87}{29}$	$2\frac{1}{2}$
22. From $\frac{63}{48}$ subtract $\frac{13}{12}$	$\frac{11}{48}$

FRACTIONAL MULTIPLICATION AND DIVISION,

INCLUDING TREATMENT OF COMPLEX FRACTIONS.

The words multiply and multiplication, by their obvious etymological associations (with *multus, much*, plural *multi, many*, and *plicāre, to fold* †) relate, in arithmetic, primarily and most properly,

* This sign, $\sim$, is employed to denote the difference of two quantities, or numerical expressions, without indicating which of them is the greater. It is seldom used, however. In the exercises in which it occurs, the fractions are to be reduced to others having a common denominator, and the less is to be taken from the greater.

† The Latin verb *multiplico, multiplicare, to multiply*, appears to have been formed from, and is certainly associated with, the adjective *multiplex*, which further is composed of *multus* and the root *plic*, found in *plicāre* and *πλέκειν, to fold*. The root *plic* appears to have related chiefly to the notion of folding, in the sense of bending and creasing. But, from the notion of bending and creasing, it appears that, by a very natural transition, people's minds passed to the notion of repetition, or of contemporaneous or successive presentments of equal quantities or magnitudes, or of like objects generally; and even to the

to the making manifold, or the taking of any kind of object, or group of objects, *many times*. When a group of single objects is to be taken many times, or taken manifold, or multiplied, there is usually the requirement to find, in the ordinary decimal system of numeration, how many of the single objects will result from the taking of the group the stated number of times.

Thus, if the group consists of six objects, and if we are to take it 4 times, we say—

four times six are twenty-four,

and so we multiply 6 by 4, and get twenty-four.

We can likewise say—

five times six are thirty,
and six times six are thirty-six,

and, obviously, we shall be continuing to perform the same kind of process if, proceeding to smaller numbers of times, we go on to take anything three times, or two times; as, for instance, when we say—

three times six are eighteen,
and two times six, or twice six, are twelve.

Now, although the designation *multus*, or *many*, would ordinarily not be applied to such small numbers as two or three, yet, from the perfect similarity of the process in taking two or three times a group to the process of taking nine or ten times, or fifty times, the group, the name multiplication comes to be extended to all such cases alike, and no important difficulty, or no misleading tendency, is found to be involved in this slight extension of scope of the term *multi* in the word multiply.

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presentment of a single one of such quantities, magnitudes, or other objects. This is clearly indicated in the words *simplex*, *duplex*, *triplex*, and *quadruplex*. Thus:—

"Illi robur et æs triplex
Circa pectus erat, qui fragilem truci
Commisit pelago ratem
Primus."—Horace, *Odes*, Book I. Ode 3—

which may be translated:—That man had oak and triple brass around his breast, who first entrusted a frail bark to the wild sea.—From this and numerous other examples which might readily be cited, it is clear that, in the times of Horace, Ovid, and Virgil, the term *plex*, in such words as *simplex*, *duplex*, *triplex*, *quadruplex*, and *multiplex*, did not mean a crease or bend, but that the notion conveyed by it was the successive or contemporaneous presentment of equal or mutually resembling objects, or the presentment of one single object considered as disassociated from others, as in the case of *simplex*. In the foregoing quotation the term *plex* related to successive *sheets* or *laminæ* of brass piled or adapted (*applied*) one on another. This is the same employment of the term as when people nowadays speak of one ply, two ply, or three ply of blanket on a bed, quite irrespective of any notion of folding in the sense of creasing, which may or may not exist in the arrangement of the bed clothes. The objects successively or contemporaneously presented were not necessarily such as would admit of bending or creasing. Thus we have *triplex cuspis* in Ovid, meaning Neptune's trident; and *triplices sorores* in Ovid, meaning the three sisters, the Fates; and we have in Plautus *pecuniam quadruplicem abs te auferam*, meaning, I will take from thee a fourfold quantity of money.

Each of these cases hitherto has been a case of multiplying by an integer, or proper number. We cannot, however, well go on retaining the same language for employment in reference to certain fractional operations, which, from their having a kind of sameness in character with those integral multiplications, are commonly called multiplication by fractions and multiplication by mixed numbers. Thus we cannot well go on to speak of multiplying 6 by $4\frac{2}{3}$, and multiplying 6 by $\frac{1}{3}$, and to say that

four-and-two-thirds times six are twenty-eight,

and, worse still, to say that

$\frac{1}{3}$ times (or time) six are, or is, two.

The language which suits perfectly well for multiplication by a proper number fails entirely for a closely corresponding operation by a fractional numeric; and yet the fractional operation contemplated in the extension of the language is a real and true and a very important operation in arithmetic.

As already mentioned, it is commonly spoken of as *multiplication by a fraction, or by a mixed number*; or it may likewise, but more briefly, be called *fractional multiplication.*

In other language, however, alike for both the fractional and the integral operation, both varieties of like operations may be well and clearly expressed. Thus we may say:—

$$\left\{\begin{array}{l}\text{One-third of 6 is 2}\\ \text{or,}\quad \tfrac{1}{3}\text{ of 6 is 2}\end{array}\right\}$$

$$\left\{\begin{array}{l}\text{One-and-a-half of 6 is 9}\\ \text{or,}\quad 1\tfrac{1}{2}\text{ of 6 is 9}\end{array}\right\}$$

$$\left\{\begin{array}{l}\text{Four-and-two-thirds of 6 is* 28}\\ \text{or,}\quad 4\tfrac{2}{3}\text{ of 6 is 28}\end{array}\right\}$$

$$\begin{array}{lr}\text{4 of 6 is*} & 24\\ \tfrac{2}{3}\text{ of 6 is} & 4\\ \text{3 of 6 is} & 18\\ \text{2 of 6 is} & 12\end{array}$$

The various processes, fractional and integral, exemplified in these statements do really coalesce harmoniously together as merely varying gradations of one common process—a process wider in scope than multiplication properly so called, and including true multiplication within it; but which, in arithmetic, and in algebra, and in the higher branches of mathematics generally, for want of a better name, has come to be spoken of under the name *multiplication.*

It is, however, difficult, or impossible, to realise any clear notion of the meaning of the true process of taking a third of 6 and getting 2, through speaking and thinking of that as a process of *multiplying*

* It is a matter of no importance in principle, whether we use the word *is* or *are* in such cases as these. Thus we may quite allowably say three of six *is* 18, instead of three of six *are* 18; and we may say $\frac{1}{3}$ of 6 *is* 2, rather than $\frac{1}{3}$ of 6 *are* 2. Often, indeed, in fractional arithmetic the singular suits rather better than the plural in such cases. (See also foot note, page 74.)

6 by $\frac{1}{3}$ and getting 2, a number which, instead of being manifold of 6, is less than once 6. The difficulty, which is only one of language, misleading or wrongly suggestive, would be abated, and at the same time an advantage in brevity would be gained, if we would prune away the injurious member *multi* from the word multiply, and say *ply* and *plication* instead of *multiply* and *multiplication*. The long names must of course be retained for ordinary use in accordance with established practice; but already it may be good to have the simplified names ply and plication available for occasional use, as enabling us to give easy and clear explanations, and to avoid the perplexing and often seemingly irrational statements which have to be made in fractional arithmetic under the nomenclature based on the word multiplication. No new confusion or ambiguity of any importance would be introduced by the proposed modification. The terms *to ply* and *plication* as thus derived for service in arithmetic, with precise significance attached to them, do not obtrude on the mind, through other etymological associations, or through other usages, any notions contradictory to, or rudely conflicting with, the meanings which they would be so employed to convey. In dropping the injurious part out of the compound word multiplication, we would retain the fairly expressive part *plication* in quite as good a sense as before for the useful purposes it has been made to serve. Some further remarks and suggestions on this subject of arithmetical nomenclature and expression will be given in connection with the explanations on "division," integral and fractional, which will now be entered on.

The name *division*, as applied to operations in arithmetic, has, alike with the name *multiplication*, come to be extended to include operations very different from any which it can obviously suggest, or with which it has any reasonable connection. Through this it occurs that when we proceed to treat of the operations which are called *fractional division*, or *division by a fractional divisor*, the best course to take is to use the word *division* merely as a name whose special significance for this matter is to be assigned by definition, but is not to be sought for by any consideration of the ordinary meaning of the word in common language as relating to the dividing of something into parts, whether equal or unequal. We may, however, very reasonably make a commencement by considering some cases in which the name *division* really has an obvious rational connection with the operation which it is used to name. The name is already established in use, and we must continue to employ it, and so we may well do it the fullest justice possible by noticing the cases to which it has some real or good adaptation.

In the chapter of Introductory Explanations for Division (commencing at page 31) it has been shown, in effect, that dividing by 4 in arithmetic means, in one of its senses at least, the dividing of something (which might be a quantity, a group of single objects, or a number abstractly considered) into 4 equal parts, and finding what any one of those parts will be; also that dividing by 5, in like manner, means the dividing of something into 5 equal parts, and finding what any one of those parts will be. But if, instead of

taking either the integral divisor 4 or the integral divisor 5, we take, for instance, the fractional numeric $4\frac{1}{3}$, which is between 4 and 5, and if we try to find some operation on the original thing (commonly called the dividend), which operation shall be related to $4\frac{1}{3}$ in a manner somehow corresponding to that in which the other two operations were related the one to 4 and the other to 5, we see at once that the previous mode of stating the nature of the required operation as being *division into equal parts* fails for this modified case. We can, however, find a corresponding operation which we may still truly represent as being division, although *not into equal parts*. We may state the requirement for the new case which may be adduced, as being to divide something into four parts mutually equal, and one part equal to $\frac{1}{3}$ of one of the equal parts, and then to take as the required result one of the equal parts so found. In the first case we divide into four equal parts, in the second we divide into five equal parts; but it would be too violent an overstraining of language, and would too much involve perplexity of thought, to say for the third case that in it we divide into four-and-a-third equal parts. The mode of thought and language with which we have started fails to connect the ordinary process called division by an integer with the allied fractional operation somehow perfectly resembling it, so as to represent them consistently as harmonious gradations of one general process, which better views show them really to be.

When we come further to do what is called *dividing by a fraction less than unity*, the language and mode of thought which have been entered on through considering cases of dividing into equal parts, and have been extended to the general subject spoken of as *Division*, come to be more unsuitable still; because, for instance, under the ordinary language in established use, we are compelled to say that to divide anything by $\frac{1}{3}$ is to take 3 times the thing, and to divide anything by $\frac{1}{5}$ is to take 5 times the thing—that is, in each case, not to take a portion of it obtained by dividing it into parts, but to take *more than the whole*. The language here does not admit of elucidation; and the less it is discussed before the incipient student, the less is his mind likely to be perplexed in a subject which otherwise admits of being made easy and clear.

Now, to pass to clear and consistent modes of thought, while retaining the word "division" for use in its ordinary significance in arithmetic, but employing it as a mere name without regard to its mode of introduction, we may notice that in "dividing" by 4, or taking the fourth part of a thing (which may be a quantity, a group, or an abstract numeric), we are finding the thing of which, if 4 be taken, the result will be equal to the original thing. So in fixing what the name "division" is to mean, or what meanings it is to have, we may go on in like manner to say that in "dividing" anything by $4\frac{1}{3}$ we are to be finding the thing of which, if $4\frac{1}{3}$ be taken, the result will be equal to the original thing.

In saying this we are stating very clearly what is really meant in arithmetic, and in other branches of mathematics, by dividing by

$4\frac{1}{2}$; and we are indicating or suggesting what is meant in general by dividing by any numeric whatever, whether integral or fractional.

Thus, going forward in exactly the same way, we say that in "dividing" anything by $\frac{1}{3}$ we are to be finding the thing of which, if $\frac{1}{3}$ be taken, the result will be equal to the original thing. Likewise we say that in dividing anything by $\frac{2}{3}$ we are to be finding the thing of which, if $\frac{2}{3}$ be taken, the result will be equal to the original thing.

From what has already been explained, it will be noticed that in the process called *division* there are three terms concerned. One is called the *dividend*, another is called the *divisor*, and the third may be called the *requisite*. Using these names, we may define or describe the process of division by any numeric* thus:—

DEFINITION OF DIVISION BY ANY NUMERIC.—To divide anything called the *dividend*, which may be a quantity, a group of single objects, or a numeric, by any numeric called a *divisor*, is to find another quantity, group, or numeric, called the *requisite* or the *resultant*, such that if the divisor numeric were taken of that requisite, the dividend would be brought back again.

For instance, to divide the quantity 20 lbs. by the numeric $\frac{4}{5}$, will consist in finding for result a requisite such that $\frac{4}{5}$ of it will be 20 lbs. The requisite here, when found by any process, must come out to be 25 lbs., as will be seen by observing that if $\frac{4}{5}$, which is the divisor numeric, be taken of 25 lbs., 20 lbs. will be obtained, and that is the dividend brought back again; and so, according to our definition, any process for finding 25 lbs. as the requisite, is called division; and in this case it is division by a fractional numeric, $\frac{4}{5}$, which is less than unity, and which therefore makes the requisite be greater than the dividend.

Divisor.	Dividend.	Requisite.
$\frac{4}{5}$)	20 lbs.	(25 lbs.

Thus we see that, in ordinary language, to "divide" a thing called the "dividend" by $\frac{4}{5}$ means to do that which would be reversed or undone by "multiplying" the requisite result by the "divisor"; or,

* It is to be noticed that, in accordance with what is taught in the Introductory Explanations for Division, at pages 34 and 35, there is another kind of operation, called also *Division*, in which a quantity of some kind of thing is said to be divided by another quantity of the same kind of thing, when the result sought is commonly called a *quotient*, and is also commonly called *a number, whole or fractional, of times*. In this way it is said that if we divide £6 - 4 - 3 by £2 - 1 - 5, we get as result the quotient 3 *times*, which means that the quantity called the divisor is contained in, or may be taken out of, the quantity called the dividend three times, or that 3 of the divisor would be a quantity equal to the dividend. In the same way it is said that if we divide a length of 3 yards by a length of 2 feet, we get as quotient $4\frac{1}{2}$ *times*. This is often spoken of as meaning that a length of 2 feet can be taken $4\frac{1}{2}$ times out of a length of 3 yards; but it would be better spoken of by saying that the length of 3 yards is $4\frac{1}{2}$ of the length of 2 feet. This kind of operation, which is one of the different kinds that are ordinarily included under the name division, and is spoken of as division of a quantity by a like quantity, or is in fact the finding of the *ratio* (see page 93) of the quantity called the dividend to that called the divisor, is to be noticed as NOT being included within the operation which in the text here is referred to under the designation of *division by a numeric*.

what comes to the same, we see that it is exactly the reverse of the process of "multiplication" of something by $\frac{4}{5}$, whereby the dividend would be produced; and this something, at first unknown but required to be found, from which the multiplication process by $\frac{4}{5}$ must commence so as to end in producing the dividend, is defined as being the thing wanted, or the requisite, for the division process. But as the names "multiplication" and "division" are in their nature eminently unsuitable for describing the real processes here intended, we may, in other words, more readily intelligible, say:—If we *ply* 25 lbs. by $\frac{4}{5}$, we get 20 lbs.; and so if we reverse this process, or *disply* 20 lbs. by $\frac{4}{5}$, we get 25 lbs. Thus we notice that the process called "dividing" by a numeric, and the process called "multiplying," which must be by a numeric, are two processes, either of which, applied to the result of the other, with the same operating numeric in both cases, will undo or reverse that process, and so bring back as its own result the original term of that other process. The same may be stated in other words in saying that to ply and to disply by any numeric, the same in both cases, are two processes, either of which applied to the result of the other will exactly undo or reverse that process, and so will bring back as its own result the original term of that other process.

Out of the class of cases called *division by a numeric* it is important now to take for separate consideration the case in which not only the divisor, but the dividend also, is a numeric. For this case the previous general definition of course holds good; and, as applied to this case, it directly conveys the following particular definition:—

A Definition of Division of One Numeric by Another.—To divide one numeric called the *dividend* (or called the *primary numeric*) by another numeric called the *divisor* is to find a numeric, which may be called the *requisite* or the *resultant*, such that if the divisor numeric be taken of the resultant, the new result will be equal to the dividend.

For instance, let it be required to divide the numeric $\frac{2}{3}$ by the numeric $\frac{3}{4}$. Here, under our definition, the requisite (or resultant required) is to be a numeric such that if $\frac{3}{4}$, which is the divisor numeric, be taken of that requisite, the result obtained will be equal to the dividend, or primary numeric, $\frac{2}{3}$. Let the requisite, at present unknown, be denoted by x. Then, by what has just been said, we see that $\frac{3}{4}$ of x must be equal to $\frac{2}{3}$; or, as the same may be noted in form of an equation—

$$\tfrac{3}{4}x = \tfrac{2}{3}.$$

Multiplying both sides of this equation by 4, we get

$$3x = \tfrac{8}{3}.$$

Dividing both sides of this equation by 3, we get

$$x = \tfrac{8}{9}.$$

Thus we see that the requisite or resultant for the demanded division process is $\frac{8}{9}$.

Having got this resultant, we may test it directly according to the definition. Thus it ought to be such that $\frac{3}{4}$ of it—that is, the divisor numeric of it—shall be equal to the primary numeric $\frac{2}{3}$. Let us, then, find $\frac{3}{4}$ of $\frac{8}{9}$. We can obviously see that one-fourth of eight-ninths is two-ninths. Hence three-fourths of eight-ninths will be six-ninths, and this is obviously equal to $\frac{2}{3}$. That is to say, $\frac{3}{4}$ of $\frac{8}{9}$ is $\frac{2}{3}$; and so the numeric $\frac{8}{9}$, previously found for the requisite, is what according to the definition was wanted to be found.

As another illustration of the meaning of the definition, let us take an ordinary easy case of dividing a given whole number into a stated number of equal parts; and we shall see how the definition, intended to suit for fractional and integral division jointly, and already explained for fractional division, suits perfectly for simple integral division. Let it be required to divide the number 21 into three equal parts, or to find one of those three equal parts, or briefly to find $\frac{1}{3}$ of 21.* Here 21 is called the dividend (or the primary numeric), 3 is called the divisor; and obviously the requisite must come out to be 7, because 7 accomplishes the condition specified in the definition, that if the divisor numeric 3 be taken of 7, or if 3 of 7, or 3 times 7, be taken, the result will be 21.

Thus we see that the definition already given for the important case of division by a numeric, in which not only the divisor, but also the dividend, is a numeric, suits for fractional and integral division jointly considered, as one general process.

The case under consideration, however—that, namely, of dividing one numeric by another—may be otherwise stated by a definition next to be given, which depends upon some different courses of thought, and which for some purposes, or in some connections, will be more directly and more obviously suitable. The perfect agreement of the two different definitions will be afterwards explained.

Another Definition of Division of One Numeric by Another. (This definition will be here stated in two slightly different modes of expression, both conveying precisely the same meaning.)

(*a*.) To divide one numeric by another is to find the numeric that the one is of the other.

(*b*.) To divide one numeric called the dividend by another called

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* It will be recollected, from the teaching in the earlier part of the present chapter, that to find $\frac{1}{3}$ of 21 is the process which is commonly called to multiply 21 by $\frac{1}{3}$, or which may be called to ply 21 by $\frac{1}{3}$; but, in the passage before us here in the text, the same expression is put forward as meaning to divide 21 by 3. The state of the case is that to divide or to disply by 3 is exactly the same as to "multiply" or ply by $\frac{1}{3}$. Also, it may at this stage be mentioned that to divide or disply by $\frac{1}{3}$ is the same as to multiply or ply by 3. The two numerics 3 and $\frac{1}{3}$ are called *reciprocals*, each of the other, or either is said to be the *inverse* of the other; and to multiply or ply by a numeric is the same as to divide or disply by its reciprocal or inverse. This subject is mentioned at the present stage because attention is drawn to it by the passage above in the text, but it is here touched on only briefly and in a foot note, in order not to interrupt the course of explanations in progress in the text. More explanations bearing on the same subject will be given at various places further on in the present chapter, and in the teaching as to *inverse ratio* and *inverse proportionality*.

the divisor is to find for result the numeric which if taken of the divisor will produce the dividend.

Remark.—Hence, as will be seen by comparing the present definition with the definition of ratio given previously (p. 93), the operation of dividing one numeric by another is the same as finding the ratio of that one to that other.

For instance, let us take again an example already used in illustration of the previous definition of division of one numeric by another. Let it be required to divide $\frac{2}{3}$ by $\frac{3}{4}$. Here, under our second definition, we are to find for result the numeric that the dividend $\frac{2}{3}$ is of the divisor $\frac{3}{4}$, or the numeric which taken of $\frac{3}{4}$ produces $\frac{2}{3}$. The requisite result, when by any means found (and the mode of finding it may, for brevity, be left out of consideration here), must come out to be $\frac{8}{9}$; because $\frac{8}{9}$ is the numeric which taken of $\frac{3}{4}$ brings out $\frac{2}{3}$; or, in other words, in taking $\frac{8}{9}$ of $\frac{3}{4}$ we get $\frac{2}{3}$; a truth which may easily be proved in various ways. It may, for instance, be proved thus:—We want to find a simple expression for *eight-ninths of three-fourths*, so we may notice that one-ninth of $\frac{3}{4}$ths is $\frac{3}{36}$ths, and eight times this, or eight-ninths of $\frac{3}{4}$, will be $\frac{24}{36}$ths, which is obviously equal to $\frac{2}{3}$rds. That is, we have proved that

$$\tfrac{8}{9} \text{ of } \tfrac{3}{4} \text{ is equal to } \tfrac{2}{3},$$

and this is what was wanted to be done in order to prove that $\frac{8}{9}$ is the requisite result for the example in division under consideration.

As another illustration of the meaning of the second definition let us take again the second of the two examples already used for the first definition. Let it be required to divide 21 by 3, but let us take the requirement this time as being to find *how often* 3 is contained in 21. The answer to this question may quite properly be called a *quotient*; it is 7 *times*; and, in finding it, we have, in accordance with the second definition, been finding the numeric that the dividend (or primary numeric) 21 is of the divisor 3; or we have, also in accordance with the second definition, been finding the numeric which, if taken of the divisor 3, will produce the dividend 21. Also, in accordance with the Remark annexed to the definition, we have been finding the *ratio* of the dividend 21 to the divisor 3; a ratio which we find to be the number 7.

The perfect agreement of the two definitions of *Division of One Numeric by Another* depends on a proposition, of great fundamental importance in "multiplication," which will now be stated and established.

Proposition in Multiplication.—In respect to any two numerics, if the first numeric be taken of the second, the same result will be obtained as if the second be taken of the first; or, in other words, the second numeric *plied* by the first gives the same product as the first numeric *plied* by the second.

For example, if we assume any two numerics, $\frac{3}{5}$ and $\frac{4}{7}$; then if we take $\frac{3}{5}$ of $\frac{4}{7}$ we shall get the same result as if we take $\frac{4}{7}$ of $\frac{3}{5}$.

To show the truth of this for the particular numerics $\frac{3}{5}$ and $\frac{4}{7}$ here adduced, let us first set down

$$\tfrac{3}{5} \text{ of } \tfrac{4}{7}.$$

Then we may notice that $\frac{1}{5}$ of $\frac{4}{7}$ would be $\frac{4}{35}$, and therefore multiplying these equals by 3, we see that $\frac{3}{5}$ of $\frac{4}{7}$ is equal to $\frac{12}{35}$.

Again, if we next set down

$$\tfrac{4}{7} \text{ of } \tfrac{3}{5},$$

we may notice that $\frac{1}{7}$ of $\frac{3}{5}$ is $\frac{3}{35}$, and therefore, multiplying these equals by 4, we see that $\frac{4}{7}$ of $\frac{3}{5}$ is equal to $\frac{12}{35}$.

Thus we have proved that

$$\tfrac{3}{5} \text{ of } \tfrac{4}{7} \text{ is equal to } \tfrac{12}{35},$$

and we have proved that

$$\tfrac{4}{7} \text{ of } \tfrac{3}{5} \text{ is equal to } \tfrac{12}{35}.$$

So it follows that

$$\tfrac{3}{5} \text{ of } \tfrac{4}{7} \text{ is equal to } \tfrac{4}{7} \text{ of } \tfrac{3}{5}.$$

Like proofs might be given in all other particular cases, or the proof might easily be expressed at once in general terms by aid of algebraic notation; and so the truth of the Proposition may be regarded as established.

Remark 1. The same proposition for the case of any two proper numbers (that is, whole numerics) has been before stated and proved on page 23, to which the reader may with advantage refer; and now it is extended to numerics in general, whether whole or fractional.

Remark 2. Hence, since for any two numerics when one of them is to be plied by the other, the same result will be obtained whether the first be plied by the second, or the second be plied by the first, we may often simplify our statements by omitting to specify one of them in particular as being the plier, and the other as being the plicand; and we may speak merely of the two being plied together. In this way, in ordinary language, it is customary to speak of two numbers, whole or fractional, being multiplied together, and to designate the result as their product without considering either of them in particular as having been fixed on as the multiplier.

We can now see how, through the principle expressed in the Proposition before us, the two definitions of *division of one numeric by another* are in perfect agreement. For, under the first one, we are to find a requisite numeric such that the divisor numeric taken of the requisite will produce the dividend; and, under the second, we are to find a requisite numeric which taken of the divisor will produce the dividend; but now, by the Proposition just established, we see that the requisite numeric for accomplishing the one condition must be the same as the requisite numeric for accomplishing the other condition; or the two definitions perfectly agree in requiring one same result to be brought out.

We may also, through the principle established in the Proposition under consideration, bring out, from each of the previous

definitions of *division of one numeric by another*, a single and simplified definition equivalent alike to the one and to the other of them, while leaving out of consideration the notions which distinguish them. Thus:—

A Third Definition of Division of One Numeric by Another.—(*a*.) To divide one numeric, called the dividend or primary numeric, by another called the divisor, is to find, for result, a numeric, called the requisite or resultant numeric, such that the product of the resultant and divisor shall be equal to the primary numeric or dividend.

Or in other words, with the same meaning:—(*b*.) To divide or disply one numeric called the primary numeric by another called the displier is to find, for result, a numeric called the requisite or resultant numeric, such that the product of the requisite and the displier shall be equal to the primary numeric.

The explanations now given as to the arithmetical process called "*division by a fractional divisor*," will help to throw light on the subject of complex fractions, and that subject will next be brought under consideration.

In the Introductory Chapter on Fractions, page 49, an indication has already been given as to what is the special form of the fractional expressions which are called *complex fractions*. It was there stated to the effect that from simple fractions, having a whole number for numerator and a whole number for denominator, and having an easily intelligible meaning, we may, by obvious suggestions, be led to pass to other numerical forms or arrangements, in which, instead of the integral numerator, a fractional numeric is introduced; or in which, instead of the integral denominator, a fractional numeric is introduced; or in which any two fractional numerics are put instead of the integral numerator and integral denominator; and that the numerical arrangements so formed are called complex fractions. Thus $\frac{5\frac{1}{2}}{9}$, $\frac{8}{9\frac{2}{3}}$, and $\frac{5\frac{1}{2}}{6\frac{2}{3}}$ were offered as instances of complex fractions. It was there stated, however, to the effect that the explanations of fractions which up to that stage had been given in this treatise did not suffice to indicate what kind of meaning is to be attributed to a fractional expression having a fraction put instead of an ordinary obviously intelligible integral denominator. On that subject perfectly clear explanations can now readily be given.

To prepare the way for entering on the subject of complex fractions, it is necessary or proper first to clear away, as an impediment or stumbling-block, the notion conveyed in the common mode of designating the two terms of a fraction as *denominator* and *numerator*. These designations are perfectly well suited for the terms of a simple fraction—as, for instance, for those of $\frac{2}{3}$ or $\frac{5}{8}$. Thus, if we have to do with $\frac{5}{8}$ of an inch, we may truly and very clearly regard this length as arrived at by thinking first on the small length which may be *named* or *denominated* as one-eighth of an inch, and by then taking a length of 5 small lengths each $\frac{1}{8}$th of an inch. In this way the 8 is used as a *namer* or *denominator* for the small equal

lengths brought under consideration, and the 5 is used as a *numerator* or *numberer* telling how many of them are taken. This is a perfectly good way of thinking on and dealing with the expression $\frac{5}{8}$ of an inch, or $\frac{5}{8}$ inch. It is not the only true way, however. We may regard the expression noted by the number 5 above a line, and the number 8 below the line, with the word *inch* annexed, and standing on the whole thus, $\frac{5}{8}$ inch, as meaning *one-eighth of* 5 *inches*: we may conceive that we first take 5 inches, or multiply the unit, one inch, by 5, and that we then divide the length so got by 8, or take *one-eighth of it* for the result. In this way, the names numerator and denominator, although not so clearly expressive as in the other way, may still be regarded as not quite unsuitable. When, however, we come to treat of the length of one inch as taken 5 times, or multiplied by 5, and then of the result as being next divided or displied by $8\frac{2}{3}$, we cannot manage to force our language into making $8\frac{2}{3}$ be a *namer*, or *denominator*, of parts of an inch, of which 5 or any other number are to be taken in order to make some desired length. The nomenclature, by means of the word denominator, fails here entirely; unless, indeed, we abstain from using the word denominator in the sense of a *namer*, though retaining it as a word meaningless except by special definition.

Now, to keep clear of misleading suggestions from commonly used words, we may, for the present, occasionally name the two terms of a fraction as the *upper* and the *lower*; so that, in a simple fraction, the *lower* is the term named, and very well named indeed, the *denominator*, and the *upper* is the term named, also very well, the *numerator*. Now, we may truly define for fractional expressions, in general, whether simple or complex, the upper term as being the *plier*, or the *primary numeric*, and the lower as being the *displier* (or *divisor* in common but badly expressive language), and the entire fractional numeric as being equal to, and only a different expression for, the *resultant* of the operation of displying or dividing the upper term by the lower. And, if we take for the significance of the two members, and of the whole of a fractional numeric, this definition, in these or in any equivalent words, we are really stating or indicating very fully and truly the ordinary established significance of fractional expressions, whether simple or complex. Thus the numeric noted with 5 for upper term and 8 for lower, $\frac{5}{8}$, instead of being read or called five-eighths, may be called 5 divided by 8, or 5 displied by 8; and $\frac{5}{106}$ may be called 5 divided or displied by 106, instead of being called *five-one-hundred-and-sixths*; and likewise $\frac{5}{102}$ may be called 5 divided or displied by 102, instead of *five-one-hundred-and-twoths*, or *five-one-hundred-and-seconds*. The common language founded on the regarding of the lower term as a *namer* or *denominator*, as may be seen by the last instance, and by many others like it, often fails to suit well even when the lower term is an integer. Also, instead of attempting to read or name the expression $\dfrac{5}{8\frac{2}{3}}$ in any such way as *five-eight-and-two-thirdsths*, which would be quite unmanageable, we may call it *five-divided-by-eight-and-two-thirds*, or *five-displied-by-eight-and-two-thirds*.

It is important to notice that it is a matter of indifference as to results, though it may often be a matter of importance as to mode of consideration, and as to facility of operation, whether in taking $\frac{5}{8\frac{2}{3}}$ of any original quantity—say, for instance, of the money received for a ship sold—we multiply that original quantity by 5, and then divide (that is, disply) the result by $8\frac{2}{3}$, or whether otherwise we treat the 5 as a primary numeric, and disply (or divide) it by $8\frac{2}{3}$, so as to get a resultant numeric equivalent to $\frac{5}{8\frac{2}{3}}$, and then ply (or multiply) the original quantity—say, the price of the ship—at once by this resultant numeric. In the former mode of procedure the 5 may well be regarded as a plier, or multiplier, and the $8\frac{2}{3}$ as a displier or divisor; and, in the latter mode of procedure, the 5 may better be regarded as a primary numeric to be itself displied by $8\frac{2}{3}$, in order to obtain a resultant numeric to be used alone as a plier of the original quantity—the price, for instance, of the ship. This indicates the reason why, in a fractional numeric, especially in a complex one, it may be convenient, and may tend to clearness, sometimes to call the upper term the plier, or multiplier, and sometimes to call it the primary numeric, while in either case the lower term may be called the displier or divisor.

Some further examples in "multiplying" and in "dividing" by fractional numerics, and in simplifying and otherwise treating complex fractions, will next be worked out with full explanations; and, through the consideration of such explained examples, various important principles will be brought into notice, helping towards clear views on the subject in general, and often leading very directly to the formation of rules or instructions for easy guidance in practical work. Such rules, in this and in other parts of arithmetic, ought not to be blindly followed as substitutes for clear views of the principles concerned and of the meanings of the operations practised. The real and important utility of such rules ought to be recognised as consisting in their affording guidance for quick and easy decision as to how to proceed in practical work without the requirement on each occasion to spend time and thought in reconsidering reasons for methods of procedure previously established and understood.

Exam. 1. Let it be required to "multiply" $\frac{5}{7}$ by $\frac{2}{3}$, or, what is the same, let it be required to find $\frac{2}{3}$rds of $\frac{5}{7}$ths.

Here we may first notice that $\frac{1}{3}$ of $\frac{5}{7}$ is $\frac{5}{3\times7}$, and so $\frac{2}{3}$ of $\frac{5}{7}$ must be twice as great, or $\frac{2}{3}$ of $\frac{5}{7}$ must be $\frac{2\times5}{3\times7}$, and this is equal to $\frac{10}{21}$, which is the required product. Thus we see that:—

Rule I. To multiply one simple fraction by another, we may multiply the two numerators together to get the numerator for the required result, and multiply the two denominators together to get the denominator for that result.

The same may be brought out in another way, which for some cases is only slightly different, but for others is importantly dif-

ferent. Thus:—Let it be required to multiply $\frac{5}{7}$ by $\frac{2}{3}$, or to find a convenient expression for $\frac{2}{3}$ of $\frac{5}{7}$. The multiplier here, $\frac{2}{3}$, may be read as *a third of two*, and the whole expression may be regarded as being *a third of two of* $\frac{5}{7}$. Now 2 of $\frac{5}{7}$ is $\frac{2\times5}{7}$ or $\frac{10}{7}$; and so $\frac{1}{3}$ of this, or $\frac{1}{3}$ of 2 of $\frac{5}{7}$, is $\frac{2\times5}{3\times7}$, which is equal to $\frac{10}{21}$, and is the same as was otherwise found before.

Exam. 2. To show that in some cases there is an important distinction between the two modes of thought concerned in the two modes of procedure which have been just now adduced in Exam. 1, let it next be required to "multiply" $\frac{3}{5}$ of a score of sheep by $\frac{6}{8}$. This may be done in various ways, some much shorter than others. For one way, we may notice that a score of sheep is 20 sheep; and that $\frac{3}{5}$ of a score of sheep must be 12 sheep, because $\frac{1}{5}$ of 20 sheep is 4 sheep, and so $\frac{3}{5}$ of 20 sheep must be 12 sheep. Thus, as an equivalent for the original requirement, we may say that we are to "multiply" 12 sheep by $\frac{6}{8}$, or to find $\frac{6}{8}$ of 12 sheep. Now we cannot well in idea propose to divide a group of 12 living sheep into eight equal parts, and then to take for result 6 times one of those parts, because the group of 12 sheep is not divisible into eight equal parts, each living sheep being regarded as an essential unit, so that we cannot properly treat of an eighth of 12 living sheep as being $1\frac{1}{2}$ sheep. Twelve pounds of butter might be divided into eight equal parts, each of which would be $1\frac{1}{2}$ lb. of butter; and if we take 6 times this we shall have 9 lbs. of butter. But now to revert to the finding of $\frac{6}{8}$ of 12 sheep:—We can quite well take 6 of 12 sheep, or 6 times 12 sheep, which will be 72 sheep; and then we can take $\frac{1}{8}$ of this, which will be 9 sheep. Thus it is not in all cases a matter of indifference as to mode of thought, whether in order to "multiply" by a fraction we at first "divide" by the denominator and afterwards "multiply" by the numerator, or first "multiply" by the numerator and afterwards "divide" by the denominator. Both modes of procedure lead in the end to the same result, but it is the better way to think out the principles by courses leading to true results while not leading us to the results through the medium of nugatory expressions such as $1\frac{1}{2}$ living sheep, or $2\frac{2}{3}$ living sheep, or generally of any fraction of an essential unit. Now the question before us in the present example (Exam. 2) admits of being solved very easily by a different course. The question was to multiply $\frac{3}{5}$ of a score of sheep by $\frac{6}{8}$, or to find $\frac{6}{8}$ of $\frac{3}{5}$ of a score of sheep. We may notice that the multiplier $\frac{6}{8}$ is a fraction not in its lowest terms, and that it is equal to $\frac{3}{4}$; and so we may state the requirement as being to find $\frac{3}{4}$ of $\frac{3}{5}$ of 20 sheep. Now $\frac{3}{5}$ of 20 sheep is 12 sheep, and $\frac{3}{4}$ of 12 sheep is 9 sheep, and so the required result is 9 sheep, the same as was found before.

Again, we may proceed otherwise, thus:—

We may cease to regard the 20 sheep as a primary group to be first plied or "multiplied" by $\frac{3}{5}$, and to have the resulting group of 12 sheep afterwards to be plied or "multiplied" by $\frac{3}{4}$, whereby the resultant group of 9 sheep is arrived at. But, instead of that, we may regard 20 sheep as a primary group to be once for all plied by the frac-

tional numeric $\frac{3}{4}$ of $\frac{3}{5}$, which by RULE I. is $\frac{3\times3}{4\times5}$, and is therefore $\frac{9}{20}$. Now to ply or "multiply" the score of sheep, or 20 sheep, by $\frac{9}{20}$, we may see that $\frac{1}{20}$th of 20 sheep is 1 sheep, and $\frac{9}{20}$ths of 20 sheep must be 9 sheep, the same result as has been before found otherwise.

Again, the whole arithmetical process carried out in the solution of the question may be briefly put forward thus:—In order to find $\frac{3}{4}$ of $\frac{3}{5}$ of 20 sheep, we may at first leave all notion of the sheep out of consideration, and proceed to find $\frac{3}{4}$ of $\frac{3}{5}$ of 20—that is, to ply 20 by $\frac{3}{5}$, and then to ply the result by $\frac{3}{4}$; or, what comes to the same, it is to find what is called the product of the three numerical factors 20, and $\frac{3}{5}$, and $\frac{3}{4}$. This product, through various explanations already given, may be seen to be $\frac{3\times3\times20}{5\times4}$, or $\frac{3\times3\times20}{20}$, and this is obviously equal to 3×3, or equal to 9, because the entire expression may be regarded as $\frac{1}{20}$th of 20 times (3×3), which is simply 3×3, or is 9.

Exam. 3. Let it be required to "divide" or to disply $\frac{7}{8}$ by $\frac{3}{5}$. Here, referring to the first definition of division of one numeric by another (page 155), we have to notice that the dividend or primary numeric is $\frac{7}{8}$, and that the divisor numeric is $\frac{3}{5}$, and that we are to find a resultant numeric which is to be such that $\frac{3}{5}$ of it shall be equal to the primary numeric. Putting this briefly in the form of an equation, with the letter x taken to denote the resultant numeric required, we have

$$\frac{3}{5}x = \frac{7}{8} \qquad (1)$$

Multiplying both sides of this equation by 5, we get a new equation—

$$3x = \frac{5 \times 7}{8} \qquad (2)$$

Dividing both sides of this equation by 3, we get

$$x = \frac{5 \times 7}{3 \times 8} \qquad (3)$$

$$x = \frac{35}{24} \qquad (4)$$

Thus we have found $\frac{35}{24}$ as the requisite or resultant numeric for the demanded division or displication, because we see that this numeric $\frac{35}{24}$ is the numeric which if plied by $\frac{3}{5}$ will give our primary numeric $\frac{7}{8}$; and so if the $\frac{7}{8}$ is displied by the same operator $\frac{3}{5}$ the reverse process is performed, and $\frac{35}{24}$ is obtained as the result of the demanded displication. Now, looking back to equation (3), we see that this result has been found by inverting the divisor $\frac{3}{5}$, so as to take instead of it $\frac{5}{3}$, and then "multiplying" or plying the primary numeric by the inverse of the divisor. By consideration of this, and of any other like operations, we are led to the following rule:—

RULE II. To divide one simple fraction, called the dividend, by another called the divisor, "multiply" the dividend by the inverse of the divisor. Or, as the same may be stated in other

words:—To disply one simple fraction, called the primary, by another, called the displier, ply the primary by the inverse of the displier.

Remark.—It will be afterwards seen that to divide or disply any one numeric, called the primary, by any other numeric (whether these be simple fractions or not), we have to ply the primary by the inverse of the divisor, or displier. The rule is here only put forward for the case of simple fractions, because it is only for that case that a sufficient illustration has been here given.

A proposition of great importance will now be stated, and its truth established or illustrated by sufficient exemplification.

PROPOSITION *A*. Any numeric expressed as a fraction composed of an upper and a lower term, whether these terms be themselves integral or fractional, may without change of value have its upper and lower terms both multiplied by any one same proper number, or both divided by any one same proper number.

The truth of this proposition in part of its entire scope, including only some of its simpler cases, has already been enunciated and established in Propositions 1 and 2, pages 49 and 50, but it was not there established in its full generality. It was not established there for cases in which the given fraction is a complex one in expression, nor for the case in which, through dividing the terms of the original fraction by a number, a complex fraction comes to be produced. The truth of the proposition is now to be proved, or to be sufficiently illustrated in its full generality.

To establish the Proposition:—Let the fraction be denoted by $\frac{a}{b}$, where a and b may represent any numerics whole or fractional. Now, referring to the definition of meaning of a fraction in general, given on page 160 (where it is stated to the effect that the entire fractional expression is by definition made to signify, in value, the resultant of the operation of displying or dividing the upper term by the lower), and using along with that any one of the three definitions given on pages 155, 156, and 159 of division of one numeric by another, but preferably the third of them, we may see that if we put x to denote the value of the fraction, irrespective of any particular numerical arrangement for expressing it, the value x must be the numeric, which is such that

$$x \times b = a \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

Now, if the terms of the original fraction be both multiplied by 5, the fraction is changed to $\frac{5a}{5b}$, and if x' be put to denote the value of this, we have likewise under the same definition

$$x' \times 5b = 5a,$$

from which it follows obviously that

$$x' \times b = a \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

Comparing this with equation (1), we see that x' must be equal to

x. So we see that the original fraction $\frac{a}{b}$ may, without change of value, have its upper and lower terms both multiplied by the number 5, the fraction being thus changed in expression to $\frac{5a}{5b}$ without change of value. A like proof could obviously be given for any other whole number whatever taken as multiplier for both terms of the fraction; and so the part of the Proposition relative to multiplying both terms by any one integral numeric whatever is now proved.

Next reserving for use formula (1) as already brought out, we may proceed to notice that if the terms of the original fraction be both divided by 3 the fraction is changed to $\frac{\frac{a}{3}}{\frac{b}{3}}$; and if now x'' be put for the value of this, we have by the same definition as before

$$x'' \times \frac{b}{3} = \frac{a}{3} \text{ .... } (3)$$

from which it follows obviously that

$$x'' \times b = a.$$

Comparing this with equation (1), we see that x'' must be equal to x. So we see that the original fraction $\frac{a}{b}$ may, without change of value, have its upper and lower terms both divided by the number 3, the fraction being thus changed in expression to $\frac{\frac{a}{3}}{\frac{b}{3}}$ without change of value. A like proof could obviously be given for any other whole number whatever taken as divisor for both terms of the fraction, and so the part of the Proposition relative to dividing both terms by any one same number whatever is proved. But the other part of the Proposition was proved before, so now the whole Proposition is proved.

The foregoing Proposition leads very directly to an easy process for simplifying complex fractions. Thus:—

Exam. 4. Let it be required to find a simple expression for the complex fraction $\frac{\frac{7}{8}}{\frac{3}{5}}$, which may be read as *seven-eighths divided by three-fifths.* It may be noticed that this requirement to simplify the given complex fraction is really essentially the same as the requirement in the preceding example (Exam. 3), to "divide" or to disply $\frac{7}{8}$ by $\frac{3}{5}$; but now using our Proposition *A*, we may proceed under a different mode of thought. Commencing with the given fraction, we may multiply its upper and lower terms by 8, which is chosen as being the lower term of the subordinate fraction constituting one of the terms of the entire fraction. The upper term $\frac{7}{8}$ on

being multiplied by 8 becomes 7, and the lower term $\frac{3}{5}$ on being multiplied by 8 becomes $\frac{8 \times 3}{5}$; and so the entire fraction, through having its upper and lower terms both multiplied by the number 8, becomes without change of value $\frac{7}{\frac{8\times 3}{5}}$. Next let us multiply the upper and lower terms of this modified expression by 5, which is chosen as being the lower term of the one term yet remaining fractional in the entire fraction. The upper term 7, on being multiplied by 5, becomes 5×7; and the lower term $\frac{8\times 3}{5}$, on being multiplied by 5, becomes 8×3; and so the entire fraction, without change of value, becomes $\frac{5\times 7}{8\times 3}$; and this is obviously $\frac{35}{24}$.

From this we may obviously draw the following rule, which may be seen to be really only a repetition of Rule II. in slightly altered words, and arrived at by a different mode of thought:—

Rule III. To simplify a complex fraction having a simple fraction for its upper term and a simple fraction for its lower term; invert the lower term and multiply the upper by it. The product will be a simple fraction equal to the original fraction. This, if not in its lowest terms, may usually be required to be reduced to its lowest terms, so as to constitute the demanded result.

Another important proposition, one which is more extensive in scope than either of the two parts of Prop. *A*, may be brought out by a combination of those two parts, through consideration of the performance, first of the operation referred to in one of the two parts on a given fraction, and then of the operation referred to in the other on the result of the first; and by observing that as neither of these two operations alters the value of the fraction on which it is performed, the two in succession will not change the value of the original fraction. So it may readily be seen that—

Proposition *B*. Any numeric expressed as a fraction composed of an upper and a lower term, whether these terms be themselves integral or fractional, may without change of value have its upper and lower terms both "multiplied" (plied) by any simple fraction whatever.

Or further, since by various processes, such as are in course of being explained, any given numeric whatever may be reduced in form to a simple fraction, it will follow that the statement in the Proposition need not be restricted to the "multiplication" of the upper and lower terms by a simple fraction, but that the Proposition may be stated in the following modified form:—

Proposition *C*. Any numeric expressed as a fraction composed of an upper and a lower term, whether these terms be themselves integral or fractional, may without change of value have its upper and lower terms both "multiplied" (plied) by any numeric whatever.

For illustration of what is stated in Prop. *B* we may take the following example:—

Exam. 5. Let it be required to simplify the complex fraction $\frac{5}{7 \div \frac{2}{3}}$.

Here the lower term is $7 \div \frac{2}{3}$, and we may notice that if we "multiply" this by $\frac{2}{3}$ we shall undo or reverse the division by $\frac{2}{3}$, and we shall leave 7 for the lower term for the entire fraction; and if we multiply also the upper term 5 by the same fractional numeric $\frac{2}{3}$ we shall get, instead of the entire original numeric, without change of value, $\frac{\frac{2}{3} \times 5}{7}$, $\frac{\frac{10}{3}}{7}$, or $\frac{1}{7}$th of $\frac{10}{3}$, or $\frac{10}{21}$.

Remark.—The generalised expression in Prop. *C* of what was stated in a less general way in Prop. *B* comes more into service for facilitating operations in arithmetic under algebraic expression than in ordinary arithmetic with direct expression in numerical figures. The generalized statement is offered here chiefly for the purpose of preventing the reception of any impression from Prop. *A* and Prop. *B* to the effect that the numeric which may be used as multiplier for the upper and lower terms should necessarily be an integer or a simple fraction. It is to be observed also that the proposition is more readily brought out and illustrated for the cases put forward in Prop. *A* and Prop. *B*—that is, for the cases when the multiplier for the upper and lower terms is an integer or a simple fraction—than for cases in general as comprised in Prop. *C*, when the multiplier is allowed to be any given numeric, whole or fractional, and anyhow expressed.

Inverses or Reciprocals of Numerics.—It may be recollected that, according to the explanations given in Exam. 3, page 163, together with the statements in Rule II. on the same page, a fraction is said to be *inverted* when its upper and lower terms are interchanged to derive a new one from it; and that the new one may be called the *inverse* of the old. It follows obviously that either of them is the *inverse* of the other. Either of them is also very commonly called the *reciprocal* of the other: or the two regarded in their mutual relation as a pair are called *reciprocals.*

The subject of inverses or reciprocals may also well be entered on in a slightly different way, by starting at first with the following definition, with which all that has been already stated on the subject will be found to be in perfect accordance:—

Definition. (1.) The inverse or reciprocal of any numeric is the numeric which is such that "multiplication" by it, performed on the result of a "multiplication" by the original numeric, will undo or reverse that "multiplication." Or, in other words, which will be found to be equivalent, we may say that—(2.) The inverse or reciprocal of any numeric is the numeric which is such that if the original and it be "multiplied" together as factors, the product will be unity.

A few examples will make this easily intelligible.

Exam. 6. If we have first under consideration $\frac{3}{5}$ of the rent of a farm, we are commencing with the result of a "multiplication" of the rent by $\frac{3}{5}$. Now, obviously we shall undo or reverse that original

"multiplication" if we "multiply" this result by $\frac{5}{3}$; as, by so doing, we get $\frac{5}{3}$ of ($\frac{3}{5}$ of the rent), which is obviously equal to the rent; and in coming back again to the rent, from which the original process had commenced, we are in effect undoing or reversing that process; and so, in accordance with the first expression of the definition, $\frac{5}{3}$ is the inverse or reciprocal of $\frac{3}{5}$.

Next, referring to the second expression of the definition, we may notice that, without any reference to the rent in particular, it is true in general that $\frac{5}{3}$ of $\frac{3}{5}$ of anything duly divisible is equal to the thing itself, or that $\frac{5}{3}$ of $\frac{3}{5}$, or $\frac{3}{5} \times \frac{5}{3}$, is equal to 1; and so, under the second expression of the definition also, each of the numerics $\frac{3}{5}$ and $\frac{5}{3}$ is the inverse or reciprocal of the other.

Exam. 7. Since 6 times $\frac{1}{6}$ of the rent would be the rent, we see that the two numerics 6 and $\frac{1}{6}$ are reciprocals each of the other; or, in other words, we may say that each of them is the reciprocal or the inverse of the other.

Exam. 8. It may readily be seen likewise that the reciprocal or inverse of $\dfrac{3\frac{2}{5}}{6\frac{7}{8}}$ is $\dfrac{6\frac{7}{8}}{3\frac{2}{5}}$. An easy way to illustrate this is by means of the second expression of the definition. Thus if we multiply these two numerics together, we get $\dfrac{3\frac{2}{5} \times 6\frac{7}{8}}{6\frac{7}{8} \times 3\frac{2}{5}}$, and this product, having its upper and lower terms equal, must be itself unity, and so the two numerics are each of them the reciprocal or inverse of the other.

Some further examples will next be given, in which various principles and methods will be illustrated, and in which operations in "division" by numerics will come specially to be performed and explained.

Exam. 9, part 1. If 5 lbs. of copper be melted with 3 lbs. of tin, how much copper will there be per one pound of tin?

Obviously there will be $\frac{1}{3}$ of 5 lbs. of copper to $\frac{1}{3}$ of 3 pounds of tin; or, what is the same, there will be $\frac{5}{3}$ of a pound of copper to 1 pound of tin.

This very easy case will help in preparing for others not so simple. We may notice that we have to commence with a given *rate* of copper to tin,

5 lbs. copper *per* 3 lbs. tin;

and the requirement is, to find an equivalent rate in terms of

a requisite quantity of copper *per* 1 lb. of tin.

So we may see that if we divide both terms of the given rate by the *number* of pounds in its term which is to be changed to 1 pound, we shall get for the required equivalent rate

$\frac{1}{3}$ of 5 lbs. copper *per* $\frac{1}{3}$ of 3 lbs. tin;

or, $\frac{5}{3}$ lb. copper *per* 1 lb. tin.

Or, briefly, to get the requisite quantity of copper which will form one term for an equivalent rate with 1 lb. of tin for the other term, we have to divide the quantity of copper stated as one term of the

given rate by the *number* which in the given rate expresses in pounds the quantity of tin which is the other term of that rate.

Exam. 9, part 2. Next, if 5 lbs. of copper be melted with $\frac{3}{4}$ lb. of tin, how much copper would there be per pound of tin at the same rate?

Here, as 1 pound of tin is $\frac{4}{3}$ of ($\frac{3}{4}$ of a pound of tin), the copper that would go to a pound of tin would be $\frac{4}{3}$ of the copper that goes to $\frac{3}{4}$ of a pound of tin; or the copper per pound of tin must be $\frac{4}{3}$ of 5 lbs. But, in taking $\frac{4}{3}$ of 5 lbs., or, as it is called, in "multiplying" 5 lbs. by $\frac{4}{3}$, we are doing what is called dividing 5 lbs. by $\frac{3}{4}$—that is, by the originally stated numeric, of which this $\frac{4}{3}$ is the inverse. So we see that whether the numeric expressing in the pound unit the quantity of tin be a proper number, or a fraction less than unity, in either case to find the quantity of copper going at the same rate to the pound of tin, we have to "divide" the stated quantity of copper by the *numeric* expressing in the pound unit the stated quantity of tin.

Exam. 9, part 3. Further, if 5 lbs. of copper be melted with $4\frac{1}{4}$ lbs. of tin, how much copper will there be per one pound of tin?

Here $4\frac{1}{4}$ lbs. of tin may be expressed as $\frac{17}{4}$ lb. of tin; and as 1 lb. of tin is $\frac{4}{17}$ of ($\frac{17}{4}$ lb. of tin), the copper going to 1 lb. of tin must be $\frac{4}{17}$ of the copper going to $\frac{17}{4}$ lb. of tin; or the copper going to one pound of tin must be $\frac{4}{17}$ of 5 lbs. of copper. But, in taking $\frac{4}{17}$ of 5 lbs. of copper, or, as it is called, in "multiplying" 5 lbs. of copper by $\frac{4}{17}$, we are doing what is called "dividing" the 5 lbs. by $\frac{17}{4}$, or by $4\frac{1}{4}$. So we see that, in this case also, to find the quantity of copper that goes to a pound of tin we have to "divide" the stated quantity of copper by the *numeric* expressing in the pound unit the stated quantity of tin.

In each of the three cases already considered we have for simplicity taken the stated quantity of copper as 5 lbs.; but, whether the quantity of copper had been 5 lbs., or $2\frac{3}{8}$ lbs., or $\frac{2}{5}$ lb., or any other numeric of the pound unit, the same mode of consideration would show that the same instruction for performance of the work would still hold good; and, in all such cases, to find the quantity of copper per pound of tin the rule may be given that we are to "divide" the stated quantity of copper by the *numeric* expressing in the pound unit the stated quantity of tin.

Exam. 10. If 2*s.* 6*d.* be the cost of $6\frac{1}{2}$ lbs. of sugar, what would be the cost of 1 lb. at the same rate?

Here $6\frac{1}{2}$ lbs. sugar may be expressed as $\frac{13}{2}$ lbs. sugar; and, as 1 lb. sugar is $\frac{2}{13}$ of ($\frac{13}{2}$ lbs. sugar), the cost of 1 lb. must be $\frac{2}{13}$ of the cost of $\frac{13}{2}$ lbs.; or the cost of 1 lb. must be $\frac{2}{13}$ of 2*s.* 6*d.* But, in taking $\frac{2}{13}$ of 2*s.* 6*d.*, or in doing what is called "multiplying" 2*s.* 6*d.* by $\frac{2}{13}$, we are doing what is called "dividing" 2*s.* 6*d.* by $\frac{13}{2}$, or by $6\frac{1}{2}$.

From this, and other examples like this, we may see that in general—

RULE IV. If the cost is given for a stated quantity of anything, then, to find, at the same rate, the cost per any unit quantity of

that thing, we have to "divide" the cost for the stated quantity by the numeric expressing that stated quantity in the same unit.

The considerations which have just been adduced, in Examples 9 and 10, bring prominently into notice an important proposition, which may be stated as follows:—

PROPOSITION *D*. If the two terms of a rate be both "multiplied," or both "divided" (or, in other words, both plied, or both displied), by any one same numeric, whether integral or fractional, an equivalent rate will result.

The truth of this follows very directly from the explanations, given on pages 106 and 107, of the meaning of the word rate, and of sameness or equivalence of rate under different expressions, together with the explanations, given in the present chapter, of "multiplication" and "division" by fractional numerics; and its truth may readily be perceived by the learner through consideration of illustrations such as are offered in Examples 9 and 10.

By employment directly of this proposition taken as true, we are led into a somewhat varied mode of thought and expression for bringing out a solution of the question in Exam. 10, but the arithmetical work, on comparison with that already brought out, will be found to be virtually the same, and this agreement helps to illustrate the proposition. The varied method so arrived at is of a kind which it is good to have in readiness for frequent practical use in other cases. Thus:—

Exam. 10. *Varied Method.*—If 2*s.* 6*d.* be the cost of $6\frac{1}{2}$ lbs. of sugar, what would be the cost of 1 lb. of sugar at the same rate?

Here, dividing both terms of the stated rate by the numeric $6\frac{1}{2}$, we get by Prop. *D* as an equivalent rate—

2*s.* 6*d.* $\div$ $6\frac{1}{2}$ *per* ($6\frac{1}{2}$ lbs. sugar) $\div$ $6\frac{1}{2}$;

or, 2*s.* 6*d.* $\div$ $6\frac{1}{2}$ *per* 1 lb. sugar.

So it arises that we are to "divide" 2*s.* 6*d.* by the numeric $6\frac{1}{2}$, or to "multiply" 2*s.* 6*d.* by the inverse of $6\frac{1}{2}$. Now $6\frac{1}{2}$ may be expressed as $\frac{13}{2}$, and the inverse of this is $\frac{2}{13}$; so we have to "multiply" 2*s.* 6*d.* by $\frac{2}{13}$, and so we find the required equivalent rate to be—

$\frac{2}{13}$ of 30 pence *per* 1 pound sugar;

or, $\frac{60}{13}d.$ *per* 1 pound sugar;

or we find that the cost of 1 lb. of sugar at the stated rate is $4\frac{8}{13}d.$

Remark.—The learner should observe that when once the principles are well understood the work in this, or in other like questions, may be very briefly performed. The work in the two solutions given of Example 10, and in the three parts of the previous example, is set out at length, with explanations, in order to exemplify and illustrate various important arithmetical principles, and especially to show some practical ways in which the requisition to "divide," or disply, by a fractional numeric may arise. In practice the work, without the explanations, comes to be very short.

Exercises.

Preparatory Explanations for Group I. of the Exercises.

The first group of the exercises on the subjects of the present chapter will be confined to fractional multiplication; and, for aiding in their performance, a few preparatory special explanations and recommendations will now be given.

Rule I., as established on page 161, should be borne in mind and be well understood. That rule was—

RULE I. REPEATED. To multiply one simple fraction by another, we may multiply the two numerators together to get the numerator for the required result, and multiply the two denominators together to get the denominator for that result.

It is further to be noticed that when a multiplier or multiplicand is given in the form of a mixed number, it may be reduced to an improper fraction as a preparation for the employment of Rule I.

Also the work may sometimes more readily be performed by employment of one or both of the following rules *A* and *B* :—

RULE *A*. When the multiplicand is a mixed number, its whole number and its fraction may be separately multiplied by the multiplier, and the results may be added together to get the required result.

Example.—Thus let it be required to multiply $4\frac{3}{4}$ by $\frac{1}{2}$.

Here "multiplying" the whole number 4 by $\frac{1}{2}$, we get, as a first partial result, 2; and "multiplying" $\frac{3}{4}$ by $\frac{1}{2}$, we get as a second partial result $\frac{3}{8}$. Then adding these partial results together, we get as the final result $2\frac{3}{8}$.

RULE *B*. When the multiplier is a mixed number its whole number and its fraction may be used separately as multipliers for the multiplicand, and the two separate results may then be added together to get the required result.

Example.—Required to multiply 6 by $2\frac{1}{3}$.

Here the multiplicand is 6, and the multiplier consists of the whole number 2 and the fraction $\frac{1}{3}$. Then first multiplying 6 by the whole number 2, we get as a first partial result 12; and next multiplying 6 by the fraction $\frac{1}{3}$, we get as a second partial result 2. Then adding these partial results together, we get 14 as the final result.

Example under Rules A and B jointly.—Required to multiply $6\frac{1}{2}$ by $2\frac{1}{3}$.

Here, according to Rule *B*, we may first multiply $6\frac{1}{2}$ by 2, and then multiply $6\frac{1}{2}$ by $\frac{1}{3}$, and then add together the two partial results so obtained. So, multiplying $6\frac{1}{2}$ by 2 under Rule *A*, we get obviously $12+1$, or 13, for a first partial result; and multiplying $6\frac{1}{2}$ by $\frac{1}{3}$, we get, also under Rule *A*, $2+\frac{1}{6}$, or $2\frac{1}{6}$, for a second partial result. Then adding together these two partial results 13 and $2\frac{1}{6}$, we get for the final result $15\frac{1}{6}$.

Otherwise.—The same requirement may be executed under Rule I., without the use of Rules *A* and *B*, thus:—Required to multiply $6\frac{1}{2}$ by $2\frac{1}{3}$. We may commence by reducing the two given factors—that is, the multiplicand and multiplier—to improper fractions; and so they will become obviously $\frac{13}{2}$ and $\frac{7}{3}$. Then by Rule I. we find for product $\frac{13\times 7}{2\times 3}$, and this is obviously equal to $\frac{91}{6}$, which is equal to $15\frac{1}{6}$, which is the required result, and is the same as was before found.

Also, it may be noticed that, according to the Proposition in multiplication established on pages 157 and 158, when, of two numerics, one is to be multiplied by the other, it is a matter of indifference, as to result, which of the two be treated as the multiplicand and which as the multiplier; but sometimes one way may afford an easier process than the other; and the easier way may be selected, if noticed.

Further, it is to be noticed that if three or more numerics are to be multiplied together, it is a matter of indifference as to result in what order the multiplications be performed.

The truth of this proposition may be illustrated or proved in various ways. It may, for instance, easily be seen to follow very directly from Rule I. Thus if it be required to multiply $\frac{2}{3}$ by $\frac{5}{7}$, and then to multiply the result by $\frac{4}{9}$, we see that by Rule I. when we multiply $\frac{2}{3}$ by $\frac{5}{7}$ we get $\frac{2\times 5}{3\times 7}$ as product; and again, by the same rule, when we multiply this product by $\frac{4}{9}$ we get as an expression for the final result $\frac{2\times 5\times 4}{3\times 7\times 9}$. Now, noticing that the numerator here is the product of the three given numerators, and that the denominator is the product of the three given denominators, we may see at a glance that the same result would have been got if we had proceeded to multiply the three given numerics together in any other order. Thus, for instance, if we commence by multiplying $\frac{4}{9}$ by $\frac{2}{3}$, we get as product $\frac{4\times 2}{9\times 3}$, and if we next multiply this by the other given factor, $\frac{5}{7}$, we get $\frac{4\times 2\times 5}{9\times 3\times 7}$, which is the same result as was before found.

Also it is to be noticed that:—To multiply any

quantity (which may, for instance, be a length, an area, a sum of money, a weight of any commodity, or a period of time, or may in general be a quantity of anything variable in magnitude), by any numeric, is to find that numeric of the stated quantity.

Thus, to multiply 2*s*. 6*d*., which is a sum of money, by $\frac{1}{5}$, is to find $\frac{1}{5}$th of 2*s*. 6*d*., and the required result obviously is 6*d*.

Group I. of Exercises.

Perform the operations indicated in the following expressions.*

1. $\frac{5}{12} \times \frac{13}{17}$	14. $2\frac{9}{13} \times \frac{8}{9}$
2. $\frac{12}{13} \times \frac{7}{8}$	15. $\frac{165}{171} \times \frac{76}{135}$
3. $\frac{11}{15} \times \frac{29}{33}$	16. $\frac{25}{29} \times \frac{25}{29}$
4. $\frac{9}{14} \times \frac{3}{17}$	17. $1\frac{9}{11} \times 1\frac{37}{40}$
5. $\frac{36}{55} \times \frac{22}{23}$	18. $2\frac{9}{28} \times 1\frac{1}{65}$
6. $\frac{23}{27} \times \frac{45}{82}$	19. $\frac{5}{12}$ of $\frac{9}{20}$
7. $\frac{15}{79} \times \frac{19}{20}$	20. $\frac{5}{7}$ of $9\frac{2}{3}$
8. $19\frac{1}{8} \times 1\frac{15}{17}$	21. $\frac{3}{4}$ of $\frac{4}{5}$ of $\frac{5}{6}$ of $\frac{6}{7}$
9. $34\frac{25}{36} \times 4\frac{36}{49}$	22. $\frac{7}{8}$ of $\frac{5}{6}$ of 7
10. $21\frac{5}{6} \times 21\frac{5}{7} \times 21\frac{5}{8}$	23. $\frac{5}{9} \times \frac{7}{10} + \frac{2}{5} \times \frac{7}{12}$
11. $\frac{99}{100} \times \frac{47}{100}$	24. $\frac{7}{11} \times \frac{22}{23} \times \frac{46}{49}$
12. $\frac{8}{17} \times 1\frac{9}{11} \times \frac{4}{9}$	25. $7\frac{6}{7} \times 1\frac{4}{13} \times \frac{273}{275}$
13. $13\frac{9}{10} \times 1\frac{11}{14}$	26. $8\frac{1}{3} \times \frac{6}{7} \times 2\frac{2}{5}$

27. Multiply 4*s*. 6*d*. by $\frac{1}{2}$.

28. Find the amount of railway fares for a journey for three adults and one child, the ordinary fare being 5*s*. 4*d*. and the child's fare being charged at half the ordinary fare. This may be regarded as a case in which it is required to multiply 5*s*. 4*d*. by $3\frac{1}{2}$.

29. Find $6\frac{2}{3}$ of 12*s*. 9*d*.

30. Multiply 3 ft. 6 ins. by $\frac{1}{7}$.

31. Multiply 2 hours 20 minutes by $\frac{2}{3}$.

Preparatory Explanations for Exercises Group II.

The second group of the exercises will relate chiefly to division by a fractional numeric. Explanations bearing on this subject have been given at length in pages 152 to 159, and 163 and 164, and in some subsequent passages. These should be attended to and under-

* The answers to the exercises here, and in some other cases afterwards, will be given at the end of the volume.

stood as well as manageable at the outset; and the rule which will now be given, and which follows very readily from those explanations, will be useful.

RULE V. To divide anything called the *dividend* by any numeric called the *divisor*, multiply the dividend by the inverse of the divisor.

Exam. 1. Let it be required to divide 2*s*. 6*d*. by $\frac{1}{5}$.

Here applying the rule and observing that the inverse of $\frac{1}{5}$ is 5, we multiply 2*s*. 6*d*. by 5; and so we find 12*s*. 6*d*. for the requisite result.

Exam. 2. Required to divide 8 lbs. by $\frac{2}{3}$.

Here the inverse of the divisor is $\frac{3}{2}$, and so multiplying 8 lbs. by $\frac{3}{2}$ we get for the required result 12 lbs.

Exam. 3. To divide 15 lbs. by $\frac{2}{3}$.

Here we have to multiply 15 lbs. by $\frac{3}{2}$, which is the inverse or reciprocal of the divisor. Now $\frac{3}{2}$ of 15 lbs. is $1\frac{1}{2}$ of 15 lbs., or it is 15 lbs. + $7\frac{1}{2}$ lbs., and that is $22\frac{1}{2}$ lbs., which is the result required.

Exam. 4. To divide $\frac{3}{4}$ by $\frac{2}{5}$.

Here we have to multiply $\frac{3}{4}$ by the inverse of the divisor—that is, by $\frac{5}{2}$. So the required result may be expressed as $\frac{3}{4} \times \frac{5}{2}$, or $\frac{3 \times 5}{4 \times 2}$, which is equal to $\frac{15}{8}$ or $1\frac{7}{8}$, either of which may be given as the answer in a convenient form.

Exam. 5. To divide $2\frac{7}{8}$ by $2\frac{3}{5}$.

Here instead of $2\frac{7}{8}$ we may take its equal $\frac{23}{8}$, and instead of $2\frac{3}{5}$ we may take its equal $\frac{13}{5}$. So we have to divide $\frac{23}{8}$ by $\frac{13}{5}$; or, what is the same, we have to multiply $\frac{23}{8}$ by $\frac{5}{13}$. Thus we get $\frac{23 \times 5}{8 \times 13}$; and this is equal to $\frac{115}{104}$ or $1\frac{11}{104}$, which is the result required.

Exam. 6. To divide $12\frac{3}{4}$ by 8.

Here we have obviously to take $\frac{1}{8}$ of $12\frac{3}{4}$; and this statement is in agreement with the rule under which we are now working—Rule V.—because the inverse of 8 is $\frac{1}{8}$. Now, we may express $12\frac{3}{4}$ as $\frac{51}{4}$, and taking $\frac{1}{8}$ of this we get $\frac{51}{8 \times 4}$, or $\frac{51}{32}$, or $1\frac{19}{32}$, and either of these last two may be offered as the required result in a convenient form.

Exam. 7. Required to divide $\frac{3}{4}$ by $\frac{7}{10}$.

Here, following the directions in Rule V., we are to multiply $\frac{3}{4}$ by $\frac{10}{7}$; and so we get $\frac{3 \times 10}{4 \times 7}$, or $\frac{30}{28}$, or $\frac{15}{14}$, or $1\frac{1}{14}$, and this last may be offered as the required result in a convenient form.

Remark.—If the given dividend and divisor be fractions having a common denominator, or if, after having

been given as numerics in any other form, they be reduced to fractions having a common denominator, the common denominator may be rejected, and the numerator of the dividend may be divided by the numerator of the divisor to find the answer.

Exam. 8. Thus if it be required to divide $\frac{2}{7}$ by $\frac{6}{7}$, we may reject the common denominator 7, and set down at once $\frac{2}{6}$ or $\frac{1}{3}$ as the answer. It may be noticed that the same result would be got by application of Rule V. Thus by it we would get $\frac{2}{7} \times \frac{7}{6}$, or $\frac{2\times7}{7\times6}$, and rejecting the factor 7, which occurs both in numerator and denominator, or, what is the same, dividing both numerator and denominator by 7, we get $\frac{2}{6}$ or $\frac{1}{3}$, the same result as before.

Exam. 9. To divide $\frac{7}{100}$ by $1\frac{3}{100}$.

Here the divisor may be put into the form $\frac{103}{100}$; and so we have to divide $\frac{7}{100}$ by $\frac{103}{100}$. Now, rejecting the common denominator, we get for the requisite result $\frac{7}{103}$.

Group II. of the Exercises.

Exer. 32. Divide $4\frac{3}{4}$ by $5\frac{1}{8}$.

33. Divide $19\frac{1}{20}$ by $2\frac{1}{15}$.

34. Find a simplified expression for $\frac{13}{60} \div \frac{23}{60}$.

35. Find a simplified expression for $8\frac{2}{5} \div 8\frac{2}{3}$.

36. Divide £1 - 5 - 6 by $\frac{3}{4}$.

37. Divide 22 ft. 8 ins. by $2\frac{1}{4}$.

Carry out the processes indicated in the following expressions :—

38. $\frac{7}{9} \div \frac{15}{16}$
39. $\frac{3}{19} \div \frac{1}{54}$
40. $2\frac{16}{17} \div 2\frac{17}{18}$
41. $1\frac{7}{20} \div 1\frac{8}{55}$
42. $1 \div \frac{3}{700}$
43. $2\frac{1}{2} \div \frac{1}{1728}$
44. $7\frac{7}{9} \div 8$
45. $4\frac{1}{2} \div 15$
46. $\frac{2}{3} \div \frac{3}{11}$
47. $\frac{3}{11} \div \frac{2}{3}$
48. $\frac{3}{5}$ of $\frac{8}{9} \div \frac{6}{7}$ of $\frac{3}{4}$
49. $\frac{13}{14} \div \frac{13}{15}$
50. $\frac{7}{9} \div \frac{1}{8}$
51. $3\frac{5}{9} \div 2\frac{5}{8}$
52. $\frac{7}{22} \div \frac{113}{355}$
53. $\frac{7}{8}$ of $\frac{6}{7} \div \frac{4}{5}$ of $\frac{8}{9}$
54. $\frac{2}{3} \div \frac{5}{7} - \frac{5}{8} \div \frac{10}{11}$
55. $1\frac{3}{4} \div 2\frac{1}{2} + 5\frac{1}{2} \div 3\frac{1}{8}$
56. $365 \div 365\frac{1}{4}$
57. $7\frac{17}{49} \div 1\frac{43}{77}$

Preparatory Explanations for Exercises Group III.

This third group of exercises will be a miscellaneous set, relating to the subjects in general which have been brought forward in the present chapter on fractional multiplication and division, including treatment of complex fractions. The teaching already given throughout the chapter should be attended to as preparation for this set of exercises.

Remark.—Much saving of arithmetical labour in the bringing out of required results may often be effected by making at an early stage a scrutiny of the mutual relations of the operations at first proposed, and abandoning any operations which may be found mutually counteractive, and simplifying others if any easy means of simplification be noticed.

Exam. 1. Required to find the product of $\frac{19}{45}$, $\frac{27}{28}$, and $\frac{7}{19}$.

Here, when the proposed operations are at first indicated by signs, but not fully carried into effect, we may notice that 19 stands as a factor in both numerator and denominator, and may either be struck out, or 1 may be put instead of it in the writing down of an equal but simpler fractional expression. The 19 in the numerator is a multiplier, and the 19 in the denominator is a divisor, and the two operations by this number, 19, would be mutually counteractive, and so both may be abandoned. Again, we may notice that the factors 27 and 45 in the numerator and denominator are each divisible by 9, and so, dividing them by 9, we find that we may put instead of them 3 and 5. And, further, we may notice that, as the 7 and 28 in numerator and denominator are each divisible by 7, we may put instead of them 1 and 4. So the required product comes to be $\frac{3}{20}$. If we had at first proceeded to multiply together the factors 19, 27, and 7 in the numerator, and to multiply together the factors 45, 28, and 19 in the denominator, we would have got a true expression for the required product; but it would have been in very large numbers, $\frac{3591}{23940}$, and would have required much trouble for the reducing of it to its lowest terms. It is much easier to find modes of simplification before the factors in numerator and denominator are multiplied together than after.

$$\frac{19}{45}\times\frac{27}{28}\times\frac{7}{19}=\frac{19\times27\times7}{45\times28\times19}$$
$$=\frac{1\times3\times1}{5\times4\times1}$$
$$=\frac{3}{20},\ \textit{answ.}$$

Remark.—When mixed numbers are given or otherwise occur as factors, they are usually more easily managed by being reduced to fractional form.

Exam. 2. To find the product of $2\frac{1}{2}$, $\frac{27}{20}$, $1\frac{1}{3}$, and $\frac{1}{3\frac{1}{4}}$.

Here, reducing mixed numbers to fractional form, for $2\frac{1}{2}$ we put $\frac{5}{2}$, for $1\frac{1}{9}$ we put $\frac{10}{9}$, and for $3\frac{1}{4}$ we put $\frac{13}{4}$. Then as $\frac{1}{\frac{13}{4}}$ is the inverse of $\frac{13}{4}$, or is $\frac{4}{13}$, we use this simple fraction instead of the complex one. Then, instead of 27 and 9 in numerator and denominator, we put 3 and 1; and, instead 10 and 20, we put 1 and 2; and afterwards we drop or strike out the 4 in the numerator with 2×2 in the denominator; and we find for the required product in simple form $\frac{15}{13}$, or $1\frac{2}{13}$.

$$2\tfrac{1}{2} \times \tfrac{27}{20} \times 1\tfrac{1}{9} \times \frac{1}{3\frac{1}{4}}$$
$$= \frac{5}{2} \times \frac{27}{20} \times \frac{10}{9} \times \frac{1}{\frac{13}{4}}$$
$$= \frac{5 \times 27 \times 10 \times 4}{2 \times 20 \times 9 \times 13}$$
$$= \frac{5 \times 3 \times 1 \times 4}{2 \times 2 \times 1 \times 13}$$
$$= \frac{5 \times 3}{13}$$
$$= \frac{15}{13}, \text{ or } 1\tfrac{2}{13}, \textit{ answ.}$$

Group III. of Exercises.

Exer. 58. A sum of money, amounting to £159 - 10 - 6, is to be divided among four persons so that the first three shall receive equal shares, and that the fourth person shall receive $\frac{5}{6}$ of one of those equal shares. Find one of the three equal shares; and also find the smaller share.

The following suggestions may be helpful:—If one of the equal shares be called X, we may see that $3\frac{5}{6}$ of X will amount to the whole sum of £159 - 10 - 6. We may write this as an equation thus:—

$$3\tfrac{5}{6}X = £159 - 10 - 6.$$

Then dividing both sides of this equation by $3\frac{5}{6}$, we get—

$$X = £159 - 10 - 6 \div 3\tfrac{5}{6}.$$

So in order to find one of the equal shares we have to divide £159 - 10 - 6 by $3\frac{5}{6}$; and, to do so, we may multiply that sum of money by the inverse of $3\frac{5}{6}$.

59. If the circumference of a carriage wheel be $13\frac{1}{4}$ feet, how often will it turn in going a mile, the mile being 5280 feet?

The following suggestions may give aid:—If x be put to denote the number of turns (or rather the numeric, integral or fractional, expressing the number of turns and fraction of a turn) we have—

$$x \times 13\tfrac{1}{4} = 5280;$$

and dividing both sides of this equation by $13\frac{1}{4}$, we get—

$$x = 5280 \div 13\tfrac{1}{4}.$$

60. A safety valve, arranged so that its area exposed to the action of the steam can be very exactly ascertained, is found to expose $29\frac{2}{3}$ square inches to the steam. It is loaded with 1350 lbs. What pressure per square inch will just suffice to balance this load and make the valve be on the point of rising?

The following suggestions may be useful:—If x be the number (or rather the numeric) expressing the pressure in pounds on one

square inch, then there will be $x \times 29\frac{2}{3}$ lbs. total pressure of steam on the $29\frac{2}{3}$ square inches, and this is to be equal to 1350 lbs. Or we may note that

$$x \times 29\tfrac{2}{3} = 1350;$$

and therefore $x = 1350 \div 29\frac{2}{3}$.

61. A link in a chain of a suspension bridge has a cross-sectional area of $9\frac{3}{4}$ square inches. It is subjected to a pull of $42\frac{1}{2}$ tons. What is the pull in tons per square inch, on the supposition that the tensile stress is uniformly distributed throughout the cross-section?

The following suggestions may afford aid. We may state that there is a rate of $42\frac{1}{2}$ tons per $9\frac{3}{4}$ square inches; and then, dividing both terms of this rate by $9\frac{3}{4}$, we get as an equivalent rate $\dfrac{42\frac{1}{2}}{9\frac{3}{4}}$ tons per 1 square inch.

62. Find a sum of money such that $\frac{4}{17}$ of it shall be £89 - 14 - 6.

63. Find a sum of money such that $5\frac{2}{3}$ of it shall be £56 - 10 - 4.

64. Find a sum of money such that 5 of it shall be £77 - 3 - 5.

65. Disply £98 - 14 - 6 by $\frac{3}{17}$.

66. Ply £3 - 10 - 6 by $\frac{1}{4}$ of $3\frac{1}{2}$, and disply the result by $\frac{7}{8}$.

67. Find in a simple form the ratio of $\frac{5}{7}$ to $1\frac{3}{4}$.

68. Find in a simple form the ratio of $\frac{7}{9}$ to $1\frac{3}{4}$.

69. Find in a simple form the numeric that $\frac{3}{10}$ is of $4\frac{1}{18}$.

70. Simplify the complex fraction $\dfrac{3\frac{5}{27}}{\frac{20}{63}}$.

71. The capacity of a cistern is $62\frac{3}{8}$ gallons. A pipe supplies water to it at the rate of $2\frac{5}{6}$ gallons per minute. How long time will be required for the pipe to fill the cistern?

72. A pipe delivering water uniformly fills, in $15\frac{43}{60}$ minutes, a cistern whose capacity is known to be $26\frac{5}{8}$ gallons. What is the rate of flow in gallons per minute?

73. Find in simple form the inverse of $\frac{75}{100}$.

74. Find in simple form the reciprocal of $2\frac{3}{5}$.

75. Six persons, A, B, C, &c., have joined originally as partners all alike in the purchase of a ship, with the arrangement that the profits accruing from time to time are to be divided among them in equal shares. Afterwards B, wishing to retire from the business, sells his share in the ship to A; and C, being in want of some ready money, sells $\frac{3}{5}$ of his share to A; so that A now owns $2\frac{3}{5}$ shares in the ship. In a division of the profits, the sum paid to A, as his $2\frac{3}{5}$ shares of the profit, was £585 - 8 - 6. What was the amount of one of the equal shares of the profit on that occasion?

76. Ply 25 by $\frac{3}{5}$, and ply the result by the inverse of $2\frac{1}{10}$, and disply the new result by $\frac{3}{100}$.

77. Multiply 34 by $\frac{5}{8}$, and multiply the result by the reciprocal of $1\frac{7}{10}$, and divide the new result by $\frac{7}{1000}$.

78. What is the fraction, or what is the numeric, that 35 is of 84?

79. What is the numeric that $84\frac{3}{4}$ is of $35\frac{1}{2}$?

80. Find what is $\frac{2\frac{1}{5}}{11}$ of a score of sheep.

81. What is $\frac{2\frac{3}{4}}{11}$ of a score?

82. What is $\frac{2\frac{1}{4}}{9}$ of a score?

83. What is $\frac{2\frac{5}{6}}{17}$ dozen?

84. Five dozen eggs are packed equally into three baskets. State by number or fraction of dozen (or, in better words, state by numeric of dozen) the eggs in each basket. Next let each basket go for 5 persons. How many eggs will there be for each person?

85. Divide $2\frac{3}{4}$ by $4\frac{1}{8}$.

86. Multiply $2\frac{3}{4}$ by $4\frac{1}{8}$.

87. Divide $\frac{3}{8}$ by $\frac{5}{6}$.

88. Divide $\frac{3}{8}$ by $\frac{6}{5}$.

89. Divide $\frac{2}{3}$ by $\frac{3}{4}$.

90. Supposing gun-metal to be composed of $90\frac{1}{2}$ parts of copper to $9\frac{1}{2}$ parts of tin by weight, find how much tin (stated as a fraction of a pound) there is per pound of copper.

91. Also find how much tin and how much copper there is per pound of the gun-metal.

92. The metre is $3\frac{28}{100}$ feet British approx. Express 1 foot as a fraction of a metre.

DECIMAL FRACTIONS.

A DECIMAL FRACTION, often for brevity called a DECIMAL, is a fraction whose denominator is 10, or some number produced by the continued multiplication of 10 as factor, such as 100, 1000, &c.*

* When a number is multiplied by itself, so that it enters just twice as a factor in making a product, the product is called the second power of the number. When a number is multiplied by itself, and the product again is multiplied by the number to make a final product, so that the number enters just three times as a factor into the product, the product is called the third power of the number. In like manner the product made by taking one same number four times as a factor is called the fourth power of the number; and so on. Thus 10×10, or 100, is the second power of 10; $10 \times 10 \times 10$, or 1000, is the third power of 10; $10 \times 10 \times 10 \times 10$, or 10,000, is the fourth power of 10; and so on. Instead of the word number throughout the above statement, the word numeric might properly have been used. It is to be understood that in that statement the word "number" has been employed in the wider but less proper sense in which it is customarily often employed, being taken to mean, not only any proper number, but also any numeric, whole or fractional, greater or less than unity, or unity itself. Thus $\frac{2}{3} \times \frac{2}{3}$, or $\frac{4}{9}$, is the second power of $\frac{2}{3}$; $\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3}$, or $\frac{8}{27}$, is the third power of $\frac{2}{3}$; and so on. Further information on the subject of powers will be given in the chapter on Powers and Roots, farther on in this treatise.

Thus $\frac{7}{10}$, $\frac{9}{100}$, $\frac{3}{100000}$, $\frac{475}{100}$, and $\frac{2475}{100}$ are decimal fractions. The last two of these, it will be noticed, are numerics of the kind commonly called improper fractions, being each of them greater than unity. They might otherwise be noted as $4\frac{75}{100}$ and $24\frac{75}{100}$. In this way there is exhibited in each of them an integral part; in the one it is 4, and in the other it is 24, and in each there is exhibited a fractional part, $\frac{75}{100}$, which is less than unity. Often, in commonly used language, the entire fractional expression, though it be greater than unity, is spoken of as a *decimal fraction*, or a *decimal*; and sometimes the integral part is regarded and spoken of as a whole number or integer, and only the fractional part less than unity is called the decimal fraction. Our ordinary language is rather imperfectly expressive in this matter, not only in reference to decimal fractions but also to fractions in general. Usually, however, the meaning of the writer or speaker may be gathered sufficiently from collateral statements.

All the rules for the management of fractions in general are, of course, applicable in regard to decimal fractions.

A specially simple and easy mode of notation, however, is available for decimal fractions, which renders their management usually much easier than it would be by employment of those methods alone which are applicable to fractions in general. This mode of notation, which may be called the decimal notation as applied not only to numbers proper but to fractions, may be explained thus:—If we write down a line of consecutive figures, 542 for instance, we have the figure 2 at the right hand meaning simply 2 of whatever objects or units are reckoned, while the figure 4 next on the left means 4 tens, or 40 of them; and the figure 5 next farther towards the left means 5 groups of a hundred each, or 500 of the objects or units. We may thus notice that in passing from figure-place to figure-place in the direction from left to right, a figure in any place expresses a tenth of what the same figure would do in the place before; and we have to observe that in the usual notation for expressing a "whole number" we recognize the units place merely by its being the extreme place at the right-hand side. But, further, in order to allow of an extension of the same decimal system of notation being made, so that the extended system may serve to express decimal fractions alike with decimally grouped numbers proper, it is only necessary to provide some other way of indicating the units place, which shall not make that place necessarily be the last or terminal place in proceeding from left to right; and then to arrange that the row of figures may be extended as far as we please to the right of the units place, subject to the same convention as before, that a figure in any place shall express a tenth of what it would express in the place next on the left of it. The units place might be marked in any convenient way. It might, for instance, be indicated by a crescent marked over it. Thus the expression $54\breve{2}36$ would mean $500+40+2+\frac{3}{10}+\frac{6}{100}$, or, $542\frac{36}{100}$, or $\frac{54236}{100}$. For the purpose of suggesting readily the corresponding relations of figures equally distant to the left and to the right of the units figure, a mark placed over or under the units figure would be

very suitable. It is, however, customary, and it suits well in respect to convenience and facility in writing and printing, to mark the units figure by placing a point *just after it*, and this is called the *decimal point*. This point should always be placed above the level of the middle of the figures, to distinguish it from another point placed lower down, which is often used as an abbreviation for ×, the sign of multiplication. Most likely the use of the point in this way really had its origin in the notion, not of its being a mark specially attached to the units place for indicating that place, but of its being a separating mark, or boundary, between the "whole number" and the decimal fraction in a row of figures intended as partly integral and partly fractional. Indeed, the decimal point is often spoken of as *the separating point*, and this name suits very well, being suggestive of true notions, especially when the intention is to refer not to the units place, but to the separation between integral and fractional portions in a row of figures. It is to be particularly noticed, however, that when we wish to recognize corresponding relationships to the selected unit in figures situated to left and to right of the units place, we have to regard the corresponding figure-places as being such as are equally distant, not from the decimal point, but from the units place.

Thus, for example, if we write down in a row, as in the margin, the figure 8 several times in succession, and if we assign to any one of these figures the units place, by putting as a mark for temporary use a crescent over it, and if, as in the margin, we number alike the places equally distant to right and to left from the units place, we see that the two figures 8 on left and on right of the units place, and immediately next it, have the relation that the one on the left means 8 tens of the unit, while the one on the right means 8 tenths of the unit. Likewise we see that the 8 numbered *2* on the left means 8 hundreds of the unit, while the 8 on the right numbered *2* means 8 hundredths of the unit, and so on. There is no like relationship or correspondence between figure-places equally distant to left and to right from the decimal point.

$$\begin{array}{ccccccc} 3 & 2 & 1 & 0 & 1 & 2 & 3 \\ 8 & 8 & 8 & \breve{8} & 8 & 8 & 8 \end{array}$$

From the explanations which have now been given it will readily be seen that—

A decimal numeric is multiplied by 10, if the units place, or, what brings the same result, if the decimal point be removed one place towards the right hand; by 100, if two places; by 1000, if three places, &c.: and, conversely, a decimal numeric is divided by 10, if the point be removed one place towards the left hand; by 100, if two places; by 1000, if three places, &c.; vacant places, when there are such, being supplied in both cases by ciphers.

Thus, $\cdot 7248 \times 10 = 7\cdot 248$, or $7\frac{248}{1000}$; $6\cdot 347 \times 100 = 634\cdot 7$; $6\cdot 3 \times 1000 = 6300$. Also, $78\cdot 33 \div 10 = 7\cdot 833$; $\cdot 736 \div 100 = \cdot 00736$; $7\cdot 3 \div 100 = \cdot 073$, &c.

It also becomes evident that

The value of a decimal numeric, when its units place is fixed by the insertion of a decimal point, is not changed by annexing one or more ciphers to the end of it, nor by taking one or more away; as, in each case, the value figures retain the same positions in relation to the units figure.

Thus, $\cdot50 = \cdot5 = \cdot500 = \cdot5000$; each being equivalent to one half.

It also becomes evident that, for any given fraction, a decimal fraction may always be found either equal to it, or as nearly equal to it as we please.

Verbal Expression of Decimals.—Decimal fractions, whether fractional alone or partly integral and partly fractional, admit of being verbally expressed—that is, spoken or written in words—in several varied ways. The best way for use in all ordinary business statements and calculations, as also for all ordinary use in scientific work, as in mathematics and natural philosophy in general, and in special subjects, such as astronomy, chemistry, engineering, &c., and, indeed, the only way commonly used for reading decimal fractions from their written expression by a row of figures with decimal point, consists simply in naming the figures and the decimal point one by one in their order from left to right as they occur in the figured expression; except that often the integral portion before the decimal point may with satisfaction be verbally expressed in the ordinary way for numbers proper as taught in the chapter on Numeration and Notation, near the beginning of this treatise. Thus the decimal numeric 152·3610954 would be ordinarily, and for most purposes would be best, read as follows:—*One hundred and fifty-two, point, three six one, nought nine five, four.* In this the pauses, indicated by commas, are unimportant except as slightly facilitating the reading and copying down of a long line of figures.

There are other ways which are usually regarded as forming requisite parts of arithmetical teaching, and which are of interest as to matters of principle or of system, but are really of little if any practical use unless when the number of figures after the decimal point does not exceed about two or three. One of these may be exemplified thus:—The numeric 152·3610954 may be put into the form $152\frac{3,610,954}{10,000,000}$, the figures after the decimal point being taken as numerator for the fraction, and the denominator being 1 with the same number of noughts annexed as the number of figures thus taken for the numerator. The numeric as a whole may then be read: *One hundred and fifty-two; and three millions, six hundred and ten thousand, nine hundred and fifty-four; ten-millionths.*

Another of these ways—the one which is probably the most commonly preferred from among the systematically good but practically not useful methods at present referred to, is arrived at by carrying out to the right of the decimal point the same kind of separation of the figures by one or more groups of three, commenc-

ing from the decimal point, as is customarily done for integers having more than three figures, commencement being made from the end of the units figure where the decimal point would be situated if wanted. This may be understood from the following example :— The numeric

1,333,333·333,333,3

may be read as—One million, three hundred and thirty-three thousand, three hundred and thirty-three units, three hundred and thirty-three thousandths, three hundred and thirty-three millionths, and three ten-millionths.

As instances of a way occasionally very good for reading a decimal numeric having very few figures after the decimal point, the following may suffice:—25·31 may very well be read, *twenty-five, and thirty-one hundredths*; or *twenty-five, and three tenths, and one hundredth*:—also 45·3 may be read *forty-five, and three tenths.* Instances will next be given of a mode of reading which is rather often fallen into thoughtlessly, but which should be avoided as quite incorrect. To read 4·63 as *four point sixty three*; and to read 5·427 as *five point four hundred and twenty seven*, would be quite incorrect and should be guarded against.

Exercises in Notation and Verbal Expression of Decimals.

Write the following fractions according to the notation of decimal fractions :—

1. Three hundred and six ten-thousandths. 2. Three hundred-thousandths. 3. One ten-millionth. 4. One ten-thousandth. 5. One tenth. 6. Fifty-seven hundred-thousandths. 7. Two hundred ten-millionths. 8. Five hundred and nine-hundredths.

Express the following decimals in words :—

1.	·58	3.	·007	5.	31·73	7.	·00004	9.	·034567
2.	·106	4.	·0007	6.	3·173	8.	·00041	10.	·0000001

REDUCTION OF DECIMAL FRACTIONS.

RULE I. *To reduce a common fraction, greater or less than unity, to a decimal.* Set down, as if for a dividend for an ordinary process in simple division, the given numerator with a row of ciphers, unlimited in number, either annexed or imagined as being annexed. If any such ciphers be actually annexed, the situation where the number proper constituting the given numerator ends ought to be noticed, so as not to be forgotten, or may well be marked by insertion of a decimal point just after the units place of the number proper. Such precaution is more especially wanted in case of the given

numerator ending already with a cipher in its units place. Set down the given denominator as divisor, and proceed to divide exactly as in simple division, except that a decimal point is to be inserted just before the figure, whether a value figure or a cipher, which is the first for the finding of which any of the annexed ciphers is used. Continue the division process till there be no remainder, or till a sufficient number of figures are obtained in the resultant to afford sufficient exactitude.

Exam. 1. To reduce $\frac{19}{8}$ to a decimal.

Here, by the rule, we are to proceed as we would do in simple division if we had as dividend any number consisting of 19 followed by a row of noughts, and as divisor the number 8; and we have to characterize our resulting figures, or, in other words, we have to assign to them their local values, by inserting a decimal point just before the first figure for the obtaining of which an annexed nought is used. In the work as shown in the margin several ciphers are actually noted, not merely imagined, after the 19, and the termination of the number proper 19, given as the numerator, is marked by insertion of a decimal point between that number and the annexed noughts. Then, in carrying out the division, we find that 8 is contained in 19 twice with a remainder 3; so we set down 2 for the first figure of the result. Now, to carry on the division farther, we have to use, with the remainder 3, the first of the annexed noughts, and so we insert the required decimal point just before the place where the next figure of the result is to be put; and then dividing 30 by 8, we get 3 for that next figure, and we set it down just after the decimal point; and we have 6 as remainder. With this remainder, 6, we mentally use another of the noughts, getting 60, which we divide by 8, so getting 7 with a remainder 4. We set down this 7 for the next figure of the result; and with the new remainder, 4, we use another of the noughts, getting 40, which we divide by 8, getting 5 for the next requisite figure; and there is no remainder; and so the work is finished; and the required result complete is 2·375.

$$\begin{array}{r} 8)\underline{19{\cdot}00000} \\ 2{\cdot}375 \end{array}$$

Reason of the Process.—The process in the foregoing example has been carried out only by the rule stated, but not proved at the commencement of this chapter; and no reason has as yet been given for the process. The reason of the process may now, however, be easily and fully explained as follows:—

If to the numerator and denominator of the given fraction $\frac{19}{8}$ we annex three noughts, which is the number of annexed noughts that were actually used in the work, we get $\frac{19000}{8000}$, which must be equal to the original fraction, because the annexing of the three noughts to the upper and lower terms of the fraction is the same as multiplying each of them by 1000, and to multiply both by any one same number we know already (Prop. 1,

pages 49 and 50) does not alter the value of the fraction. Next, if we divide both terms of the fraction thus obtained by 8, we do not change the value (Prop. 2, page 50, or Prop. *A*, page 164), and so we get the same value expressed as $\frac{19000 \div 8}{1000}$. But, by the work actually performed, we have found that $19000 \div 8$ is 2375, and so our fraction may be stated as $\frac{2375}{1000}$. But the division of 2375 by 1000 here indicated will be carried out by shifting the units place three places to the left, and so making the place occupied by 2 become the units place; and this will be effected by shifting three places to the left the units mark or decimal point, which, although not written for the integer 2375, may be regarded as having its place just after the units figure 5. The shifting of the decimal point three places to the left thus gives us the decimal numeric 2·375; and this result is the same as was before brought out by the process under the rule, and it has now been brought out by a process fully explained and shown to be throughout in agreement with the one directed by the rule.

A clear abstract of the steps in the fully explained process may be brought concisely under review in the following statement:—

$$\frac{19}{8} = \frac{19000}{8000} = \frac{19000 \div 8}{1000} = \frac{2375}{1000} = 2\cdot375.$$

Exam. 2. To reduce $\frac{2}{7}$ to a decimal.

Here we proceed as if we had to divide by 7 any number consisting of 2 followed by a row of noughts; and we have to assign to the resulting figures their local values by inserting a decimal point just before the one of them which is the first for the finding of which any of the annexed noughts is used. Now we see that 7 is not contained in the number 2, which is the entire given numerator; and so with the 2 we must use the first of the annexed noughts, getting 20, to be divided by 7; but at this stage, before setting down the quotient obtainable by this subordinate division, we may timely note the decimal point just in front of the place where we intend to insert, for a beginning of the result, the figure to be so obtained. Then dividing the 20 by 7, we get 2, which we set down as the first figure for the result just after the decimal point, and we have a remainder 6. We next use this remainder with another of the noughts, getting 60; and, dividing this by 7, we get 8 with a remainder 4. We set down the 8 for the next figure of the result, and carry on the division in like manner till we have got a sufficient number of figures for the result to give all the exactitude that is judged necessary. We thus get for a very approximately exact result ·28571.

$$\begin{array}{r} 7)\underline{2\cdot00000} \\ \cdot28571 \end{array}$$

Reason of the Process.—The process in this example resembles so closely that used in the previous one that its explanation might easily enough be made out by the learner through a little reconsideration of the principles exhibited in the very full explanations already given in that previous example, and application of those

principles to the present case. The following concise statement, if considered in conjunction with those previous explanations, may be quite sufficient to make the reasons clear:—Commencing with the given fraction $\frac{2}{7}$, we have

$$\frac{2}{7}=\frac{200000}{700000}=\frac{200000\div 7}{100000}=\frac{28571\frac{3}{7}}{100000}$$

But the fraction $\frac{3}{7}$ in the numerator here is so very small comparatively to the integral part of the numerator, 28571, that it may with only very slight inexactitude be neglected, and we may take, as a very approximately true value for the given fraction, the fraction $\frac{28571}{100000}$; and as this entire expression means the hundred thousandth part of 28571, we can see that its value may be noted ·28571; because the insertion of the decimal point at a situation five grades or places to the left of where it would stand for the number 28571, reduces the local value of each figure to $\frac{1}{100000}$ of what it is in that number.

Exam. 3. To reduce $\frac{3}{512}$ to a decimal.

Here we are to proceed as we would do in simple division if we had for dividend any number expressed by 3, the given numerator, with a row of noughts following, and for division 512; and we have to assign to the resulting figures their local values by inserting a decimal point just before the one of them which is the first for the finding of which any of the annexed noughts following the given numerator, 3, is used. We first notice that 512 is not contained in 3; so, in the result, the units figure, if noted, must be 0, or its place may be left blank. Then we insert a decimal point immediately after the place regarded as the units place. Next, by using, with the 3, one annexed nought, we get 30, in which we see that 512 is not contained, and therefore we place a nought as the first figure of the requisite after the decimal point. Then we use another nought, and so instead of 30 we have 300; and, finding that the divisor 512 is not contained in this, we put another nought as the next figure of the requisite. Next again, we use another annexed nought, getting 3000; and, finding that the divisor 512 is contained in this 5 times, we set down 5 as the next figure of the requisite, and we have a remainder 440. After this the division proceeds in the usual way, a nought being annexed to each remainder: and the requisite decimal is found to be ·005859375, or $\frac{5,859,375}{1,000,000,000}$.

```
512)3·000(·005859375
     2560
      4400
      4096
       3040
       2560
        4800
        4608
         1920
         1536
          3840
          3584
           2560
           2560
           ....
```

Remark.—The fraction thus formed with 1,000,000,000, or the ninth power of 10 (see foot note, page 179), as denominator, would, by reduction to its lowest terms, be brought back to the originally

given expression $\frac{3}{512}$; and so the truth of the process already worked out would be confirmed.

Exam. 4. Reduce $\frac{5}{12}$ to a decimal.

In this example, each remainder after the first two is 8, and hence the same figure must be repeated perpetually.

$$12\overline{)500000}$$
$$\cdot 41666, \&c.$$

Exam 5. Reduce $\frac{6}{11}$ to a decimal.

Here, after two figures have been obtained in the requisite, the same series of remainders, and consequently the same series of figures in the requisite, must recur; and therefore, were the work pursued, the first two figures of the decimal would be repeated without end.

$$11\overline{)60000}$$
$$\cdot 5454, \&c.$$

A decimal for expressing a value which cannot be exactly expressed by a finite number of the decimally arranged figures is called an INTERMINATE DECIMAL; and a decimal which, by a finite set of figures, expresses exactly the value wanted is, in contrast, called TERMINATE.

When a decimal is expressed either by the continual repetition of the same figure, or of the number expressed by two or more figures, it is called a PERIODICAL or a CIRCULATING DECIMAL; and the figure, or number, so repeated is called the PERIOD.

Thus the decimals in the fourth and fifth examples are periodical, the period in the one consisting of one figure, and that in the other of two. When only one figure is repeated, the application of the terms *period* and *periodical*, though convenient, and perhaps the best on the whole, may be considered as scarcely correct, as each of the terms suggests the idea of more figures than one. On this account, some writers term such decimals *repeaters*, or *repeating decimals*.

A periodical decimal is said to be MIXED, if it consist of one or more figures prefixed to a periodical part; others are called PURE.

For brevity, in writing decimals of this kind, it is sufficient to write the period but once, and to denote its continuation by putting a dot over the first figure of the period, and another over the last; or one over the repeating figure, if there be but one figure in the period.

Thus, ·7333, &c., may be written $\cdot 7\dot{3}$, and ·5637637, &c., $\cdot 5\dot{6}3\dot{7}$.

The following considerations will explain still further the nature of these fractions. By the preceding rule we find the following results: $\frac{1}{9} = \cdot 1111$, &c.; $\frac{1}{99} = \cdot 010101$, &c.; $\frac{1}{999} = \cdot 001001$, &c.; $\frac{1}{9999} = \cdot 00010001$, &c. These decimals are all periodical; and if any

of them be multiplied by a whole number, the result will also be periodical. Thus, if we multiply the third by 128, we find $\frac{128}{999} = \cdot 128128$, &c.; if by 998, we get $\frac{998}{999} = \cdot 998998$, &c. Here we see that the value of the periodical decimal is the common fraction, whose denominator is the number expressed by as many 9's as there are figures in the period, and whose numerator is the period itself; and the same may be shown in every case. Hence we have the following rule:

Rule II. *To find the value of a pure periodical decimal:* **Take the period for numerator, and as many 9's as there are figures in the period for denominator.**

Thus, $\cdot\dot{2}9\dot{7} = \frac{297}{999} = \frac{11}{37}$; $\cdot\dot{3} = \frac{3}{9} = \frac{1}{3}$; $\cdot 999$, &c. $= \frac{9}{9} = 1$, &c.

If the decimal be mixed, its value may be easily found on the same principle. Thus, if it were required to find the common fraction which would produce the decimal $\cdot 12436436$, &c., by multiplying by 100 we should find $12\cdot\dot{4}3\dot{6}$, or $12\frac{436}{999}$. Dividing this by 100, we obtain, for the value of the proposed fraction, $\frac{12}{100} + \frac{436}{99900}$, or, by reduction to a common denominator, $\frac{12 \times 999}{99900} + \frac{436}{99900} = \frac{12424}{99900} = \frac{3106}{24975}$.

Remark here that $12 \times 999 = 12 \times (1000 - 1) = 12000 - 12$. Hence $12 \times (999 + 436) = 12436 - 12 = 12424$. A generalisation of this is obvious, and leads to the following rule:—

Rule III. *To find the value of a mixed periodical decimal:* **From the number expressed by the finite part with the period annexed, subtract the finite part for the numerator; and, for the denominator, to as many 9's as there are figures in the period annex as many ciphers as there are figures in the finite part.**

Thus, to find the value of $\cdot 8\dot{3}$, from 83 take 8, and there will remain 75, and the required fraction $\frac{75}{90}$ or $\frac{5}{6}$. In like manner, to find the value of $\cdot 26\dot{3}\dot{0}1$ we have for numerator $26301 - 26 = 26275$, and for denominator 99900. Hence the required fraction is $\frac{26275}{99900}$, or $\frac{1051}{3996}$. The values of such fractions may also be found by the summation of series, as will appear hereafter.

It may be remarked here, that when a vulgar fraction is in its lowest terms, and its denominator contains no simple factors except 2 or 5, the two prime factors of 10, the equivalent decimal is finite, but in every other case it is interminate. The cause of this is, that neither 10, 100, nor any similar number, nor a multiple of any of them, is divisible by any of the nine digits except 2 and 5.

It may also be observed, that the number of circulating figures must always be less than the units in the denominator. This is obvious from the consideration, that the number of remainders different from each other, which can arise in any operation in division, must be less than the units in the divisor. Thus, in dividing by 7, it is evident, that the only possible remainders are 1, 2, 3, 4,

5, and 6; and since, in reducing to decimals, a cipher is annexed to each remainder, there cannot be more than six dividends, and consequently six figures in the quotient all different.

Miscellaneous Exercises and Examples.

Exercises. Reduce the following fractional quantities to decimals:—

Exercises.	*Answers.*	*Exercises.*	*Answers.*
1. $\frac{7}{16}$	·4375	9. $\frac{4}{99}$	$\cdot\dot{0}\dot{4}$
2. $\frac{3}{32}$	·09375	10. $\frac{37}{79}$	$\cdot\dot{4}68354430379\dot{7}$
3. $\frac{5}{1024}$	·0048828125	11. $\frac{4}{909}$	$\cdot\dot{0}04\dot{4}$
4. $\frac{11}{40}$	·275	12. $\frac{4}{9009}$	$\cdot\dot{0}0044\dot{4}$
5. $\frac{1}{13}$	$\cdot\dot{0}7692\dot{3}$	13. $\frac{1}{41}$	$\cdot\dot{0}243\dot{9}$
6. $\frac{7}{12}$	$\cdot 58\dot{3}$	14. $\frac{1}{9}$	$\cdot\dot{1}$
7. $\frac{3}{288}$	$\cdot 0\dot{1}0489\dot{5}$	15. $\frac{1}{81}$	$\cdot\dot{0}1234567\dot{9}$
8. $\frac{6}{10001}$	$\cdot\dot{0}005999\dot{4}$	16. $\frac{1}{44}$	$\cdot 02\dot{2}\dot{7}$

Exercises.		*Answers.*
17. 10*s*. 9*d*. to the decimal of	£1	·5375
18. 0*s*. 10½*d*. ———————	£1	·04375
19. 17*s*. 7*d*. ———————	£1	$\cdot 8791\dot{6}$
20. 3 r. 11 p. ———————	an acre	·81875
21. 2 qrs. 8 lbs. ———————	a cwt.	$\cdot\dot{5}7142\dot{8}$
22. 37 perches ———————	a mile	·115625
23. 3 hours, 30 minutes ——	a day	$\cdot 1458\dot{3}$
24. 15 minutes, 30 seconds —	an hour	$\cdot 258\dot{3}$
25. 3 cwt. 1 qr. 7 lbs. ———	a ton	·165625
26. 6½*d*. ———————	a shilling	$\cdot 541\dot{6}$

Exer. 27. If the diameter of a circle be 1, the circumference is $3\frac{1}{7}$ nearly, or $3\frac{16}{113}$ more nearly. Express each of these decimally.* *Answ.* $3\cdot\dot{1}4285\dot{7}$, and 3·1415929, &c.

28 If the circumference of a circle be 1, the diameter is $\frac{7}{22}$ nearly, or $\frac{113}{355}$ more nearly. Express each of these decimally. *Answ.* $\cdot 3\dot{1}\dot{8}$ and ·318309859, &c.

29. Reduce a quarter of wheat, containing 456 lbs., to the decimal of a hundred weight. *Answ.* $4\cdot 0\dot{7}1428\dot{5}$.

30. The length of the tropical, or civil year, is 365 days, 5 hours, 48 minutes, 49·7 seconds: reduce the 5 hours, 48 min., 49·7 seconds, to the decimal of a day. *Answ.* ·2422419, nearly.

---

* The circumference, true to twenty decimal places, is 3·14159265358979323846.

Exam. 6. Required the value of $\cdot3945$ of a day in hours, &c.

Here the decimal is multiplied by 24, the number of hours in a day, and four figures being cut off towards the right, it appears that $\cdot3945$ day is equal to $9\frac{4680}{10000}$ or $9\frac{468}{1000}$ hours. The decimal $\cdot468$ is then multiplied by 60, and three figures being then cut off, there results $28\cdot080$ or $28\cdot08$ minutes. By continuing this process the value of the given decimal is found to be 9 h. 28 min. $4\cdot8$ sec.

$$\begin{array}{rl} \cdot3945 & \text{day} \\ 24 & \\ \hline 15780 & \\ 7890 & \\ \hline 9\cdot4680 & \text{hours} \\ 60 & \\ \hline 28\cdot080 & \text{minutes} \\ 60 & \\ \hline 4\cdot80 & \text{seconds.} \end{array}$$

Exam. 7. Required the value of £$\cdot5937$.

Here, by multiplying by 20, we find, that the given decimal is equivalent to $11\frac{874}{1000}$ shillings. Then, by multiplying by 12, we find $\cdot874s.$ to be equivalent to $10\cdot488$ pence. In like manner, $\cdot448d.$ is shown to be equivalent to $1\frac{952}{1000}$ farthing, or a halfpenny, very nearly; and, consequently, the required value is 11*s.* $10\frac{1}{2}d.$, nearly.

$$\begin{array}{rl} £\cdot5937 & \\ 20 & \\ \hline 11\cdot8740 & \text{shillings} \\ 12 & \\ \hline 10\cdot488 & \text{pence} \\ 4 & \\ \hline 1\cdot952 & \text{farthings.} \end{array}$$

Answ. 11*s.* $10\frac{1}{2}d.$, nearly.

The following rule is easy and useful in practice:—*To find the value of the decimal of a pound sterling to the nearest farthing:* (1.) *Take a fifth of the number expressed by the first two figures of the decimal, for the shillings of the result.* (2.) *Diminish the number expressed by the remainder with the third figure of the decimal annexed, by a twenty-fifth of itself, and what remains will be the farthings in the rest of the required value.* Thus, in finding the value of £$\cdot5937$, the fifth part of 59 is 11, the shillings of the answer; the remainder is 4, which being prefixed to 4 (the third figure increased by 1, because the next figure, 7, is greater than 5), we have 44, the twenty-fifth part of which is more nearly 2 than 1; we therefore reject 2, and have remaining 42 farthings, or $10\frac{1}{2}d.$; and hence, the answer is 11*s.* $10\frac{1}{2}d.$, nearly, the same as before. With respect to the *reason* of this process, it is evident that the given decimal is more nearly equal to $\cdot5940$ than $\cdot5930$: we use therefore the former, or its equal $\cdot594$. Now, this is equivalent to $\cdot55 + \cdot044$, the value of the former part of which would be found by the general rule for this problem, by multiplying by 20 and dividing by 100, which is the same as dividing by 5, since 20 is one fifth of 100. Then, for the value of £$\cdot044$ or £$\frac{44}{1000}$, by diminishing the denominator by one twenty-fifth part of itself, we have 960; and by diminishing the numerator by a similar part of itself, we have 42 nearly: hence £$\frac{44}{1000}$ is nearly equal to £$\frac{42}{960}$, or 42 farthings, since £$\frac{1}{960} = 1$ farthing. Practice will soon enable the learner to estimate with sufficient correctness the effect of the fourth figure of the decimal, with respect to the quantity to be rejected. In this example, instead of using 44 farthings, we might use $43\cdot7$, and it would be

easy to see that the twenty-fifth of this would be about 1·7; by rejecting which we should have 42, as before.

Exam. 8. Required the value of $\cdot 80\dot{5}$ of a yard in long measure.

This, and similar exercises, may be wrought either by converting the proposed decimal into a common fraction, in the way shown in page 186, or more easily, by employing an approximative process, as in the margin. In doing so, we carry 1 to the product of 3 and 5, because, had the decimal been continued farther, 1 must have been carried from the preceding product. For a similar reason, 7 is carried to the product of 12 and 6. The result is found to be 2 feet, $4\cdot\dot{9}$ inches, which is equivalent to 2 feet, 5 inches (see page 188). An additional 5 was annexed to the given decimal, that the result might be more distinct and certain.

$$\begin{array}{rl} \cdot 805\dot{5} & \text{of a yard} \\ 3 & \\ \hline 2\cdot 4166 & \text{feet} \\ 12 & \\ \hline 4\cdot 9999 & \text{inches.} \end{array}$$

Answ. 2 feet, 5 inches.

Exercises. Required the values of the following decimals :—

	Exercises.	*Answers.*
31.	·0675 of a cwt.	$7\frac{14}{25}$ lbs.
32.	·4625 of a ton	9 cwt. 1 qr.
33.	£·0484	$11\frac{77}{125}d.$
34.	·8845 of an acre	3 r. $21\frac{13}{25}$ p.
35.	·00213 of a day	3 minutes, $4\frac{4}{125}$ sec.
36.	£$\cdot\dot{7}$	15*s.* $6\frac{3}{4}d.$, nearly
37.	$\cdot\dot{2}8571\dot{4}$ of a cwt.	1 qr. 4 lbs.
38.	$\cdot 1\dot{1}\dot{3}6$ of a mile	36 perches, 2 yards
39.	·615 of a shilling	$7\frac{19}{50}d.$
40.	£$\cdot 48\dot{3}\dot{5}$	9*s.* 8*d.*, nearly
41.	$\cdot 238\dot{3}$ of a degree	14′ 18″
42.	£$\cdot 0\dot{6}$	1*s.* 4*d.*
43.	$\cdot 4791\dot{6}$ lb. troy	5 oz. 15 dwts.
44.	£$\cdot 4\dot{2}\dot{8}$	8*s.* $6\frac{3}{4}d.$, nearly
45.	·4375 of a shilling	$5\frac{1}{4}d.$
46.	·09375 of an acre	15 perches
47.	$\cdot\dot{4}$ foot, *long measure*	$5\frac{1}{3}$ inches

Exer. 48. The mean length of the moon's synodical revolution round the earth, that is, the time between one new or full moon and the following, is 29·530588715 days: express this in days, hours, &c. *Answ.* 29 d. 12 h. 44 m. 2·86 s.

ADDITION OF DECIMAL FRACTIONS.

RULE. (1.) Arrange the given numbers so that the separating points may all be in the same column. (2.) Find the sum as in simple addition. (3.) Point off as many places of decimals, as there are in the given number which contains most.

If any of the numbers contain periodical decimals, let these be carried out to as many places as there are in the longest of the finite decimals; or, if much accuracy be required, let them be carried as far as may be judged necessary.

In the practical application of decimals, it is known, from the nature of the particular case, to how many places the result should be true. *When a result is thus required to be true to an assigned number of places, it is proper to carry the decimals which consist of more places, to at least one place beyond the assigned number, and to reject the last figure.* In this case, it is proper to observe, that *when a decimal is not carried out to its full length, the last figure of the part retained should be increased by* 1, *if the succeeding figure be* 5, *or greater than* 5.

Exam. 1. Add together 81·4632, 9·75, and 47·388.

Here, the numbers are arranged as in the margin, and added as in common addition. The *reason* of the arrangement and operation is manifest, those figures being added together which are of the same local value.

81·4632
9·75
47·388
138·6012, *sum.*

Exam. 2. Add together $3{\cdot}\dot{7}\dot{3}$, ·873, $51{\cdot}\dot{7}$, 108·2, and 73·463128 so that the result may have four places of decimals true.

In this example, the first, third, and fifth numbers are carried to five places, each, and the last figure of the third is made 8, because the next figure would be 7. In like manner, the fifth figure of the last line is made 3, because the succeeding figure is 8. The reason of this is evident, since 30 is nearer 28 than 20 is; and 30, by the rejection of the last figure, becomes 3. In the addition, the sum of the last column is 18, from which 2 is carried, because 18 is nearer 20 than 10. The correct sum, found by carrying the decimals farther, is $238{\cdot}051279\dot{5}\dot{1}$, which by retaining only four figures of the decimal, and increasing the last of them by 1, because it is followed by 79, &c., becomes 238·0513, the same as before.

3·73737
·873
51·77778
108·2
73·46313
238·0513, *sum.*

Exer. 1. Add together 1·83, 5·674, ·3125, 18·3, 100, 38·62, 4·3957, and ·5. *Answ.* 169·6322.

2. Required the sum of 93·617843, 7·836, 12·25, ·71575, 4·391, 7·839, 3·7674285, and ·8693. *Answ.* 131·2843215.

3. Required the sum of 51·25, $3·\dot{4}$, $·6\dot{3}\dot{7}$, $7·88\dot{5}$, 7·875, $7·\dot{8}7\dot{5}$, and $11·\dot{1}$. *Answ.* $90·079\dot{3}6072\dot{4}$.

4. Required the sum of ·7354, $·735\dot{4}$, $·73\dot{5}\dot{4}$, $·\dot{7}35\dot{4}$, ·07354, and $·07\dot{3}5\dot{4}$. *Answ.* $3·08885\dot{7}99\dot{1}$.

5. Add together ·3, $·\dot{3}$, ·45, $·\dot{4}\dot{5}$, $·\dot{3}5\dot{1}$, ·6468, $·646\dot{8}$, $·64\dot{6}\dot{8}$, and $·6\dot{4}6\dot{8}$. *Answ.* $4·4766\dot{3}4567\dot{8}$.

6. Add together $·\dot{8}$, $·\dot{8}\dot{7}$, and $·\dot{8}7\dot{6}$. *Answ.* $2·\dot{6}4455\dot{3}$.

7, 8, and 9. Required the sum of the numbers given in exercises 5th, 6th, and 10th of addition of fractions, the several fractions being previously reduced to decimals. *Answ.* 6·0078125, $3·90\dot{7}1428\dot{5}$, and $1·56101\dot{1}9047\dot{6}$.

SUBTRACTION OF DECIMAL FRACTIONS.

RULE. (1.) Set the less number so that each figure in it may stand below a figure of the same local value in the greater. (2.) Then proceed as in simple subtraction, and place the separating point as in addition of decimals.

Exam. 1. From $3·\dot{5}\dot{4}$ take 1·34265.

Here, the greater number is extended, and the remainder is found to be $2·20280\dot{4}\dot{5}$.

$$\begin{array}{r} 3·5454545 \\ 1·34265 \\ \hline 2·2028045 \end{array} \textit{ diff.}$$

Exam. 2. Required the difference of 8·6 and $2·\dot{7}$.

Here, the less number is carried to four places, that the true answer may be discovered with greater certainty. In the operation, ciphers are conceived to be annexed to the greater number, and 1 is carried to the repeating figure first used, because this must have been done, had the less number been carried one place farther. The answer is found to be $5·8\dot{2}$.

$$\begin{array}{lr} \text{From} & 8·6 \\ \text{Take} & 2·7777 \\ \hline \textit{Rem.} & 5·8222 \end{array}$$

Exercises.	*Answers.*	*Exercises.*	*Answers.*
1. 3·468 − 1·2591	= 2·2089	7. $5·83 - 4·\dot{1}\dot{7}$	= $1·65\dot{8}\dot{2}$
2. $6·45 - 1·\dot{3}4\dot{5}$	= $5·10\dot{4}\dot{5}$	8. $17·\dot{4} - ·\dot{4}\dot{8}$	= $16·\dot{9}\dot{5}$
3. 56·429 − 29·6853 ...	= 26·7437	9. $3·\dot{3}4\dot{2} - 1·\dot{7}\dot{5}$ ...	= $1·\dot{5}3476\dot{6}$
4. 34·528 − 10·6347 ...	= 23·8933	10. $8\frac{3}{8} - 7\frac{3}{7}$*	= ·94642857
5. ·682 − ·09647	= ·58553	11. $7\frac{5}{72} - 4\frac{7}{65}$	= 2·9617521
6. $13·\dot{6} - 4·345$	= $9·321\dot{6}$	12. $15\frac{1}{7} - 13\frac{6}{25}$ ...	= 1·8188235

* In this exercise and the next two, the given fractions are to be reduced to decimals, and the difference taken according to the rule.

MULTIPLICATION IN DECIMAL FRACTIONS.

RULE I. Multiply the factors as in simple multiplication, and point off in the product as many places of decimals as there are in both factors; supplying the deficiency, when there is one, by prefixing ciphers.

Exam. 1. Multiply ·582 by 66·3.

Here, because there are three places of decimals in the one factor, and one in the other, there must be four places of decimals in the product.

```
    ·58 2
    66·3
   174 6
  3 492
 34 92
 38·5866, product.
```

The reason of the rule will be understood from considering, that when the denominators are supplied, the first factor becomes $\frac{582}{1000}$, and the second $66\frac{3}{10}$, or $\frac{663}{10}$; the product of which, by the rule for the multiplication of fractions (page 161), is $\frac{582\times 663}{10000}$; and hence it appears, that the product of 582 and 663 must be divided by 10000, which is effected by cutting off four figures. It is evident, that the divisor must contain as many ciphers as there are in both denominators; that is, as there are decimal figures in both factors.

Exam. 2. Multiply ·13 by ·7.

Here, a cipher must be prefixed to the product 91, as there are two places of decimals in the one factor, and one in the other.

```
 ·13
  ·7
·091
```

Exercises.	*Answers.*	*Exercises.*	*Answers.*
1. ·78 × ·42	= ·3276	7. ·1 × ·1 × ·1 × ·1 ...	= ·0001
2. 7·8 × 4·2	= 32·76	8. 13·825 × 5·128 ...	= 70·8946
3. 7·49 × 63·1	= 472·619	9. ·08 × ·036	= ·00288
4. ·144 × ·144	= ·020736	10. ·31 × ·32...........	= ·0992
5. 1·05 × 1·05 × 1·05	= 1·157625	11. 3·18 × 41·7	= 132·606
6. 36·48 × ·475	= 17·328	12. 62·38 × 7	= 436·66

When the number of decimal figures is great, or the factors numerous, the figures obtained by the preceding rule are, in many cases, unnecessarily and inconveniently numerous. The following approximate rule will be found very useful in such cases.

RULE II. (1.) Count off, after the separating point in the multiplicand (annexing ciphers, if requisite), as many figures of decimals as it is necessary to have in the product. (2.) Below the last of these, write the unit figure of the multiplier, and write its other figures

in reversed order. (3.) Then multiply by each figure of the multiplier, thus inverted, neglecting all the figures of the multiplicand to the right of that figure, except to find what is to be carried, and let all the partial products be so arranged, that their right-hand figures may stand in the same column. (4.) Lastly, from the sum of these partial products, cut the assigned number of decimal places.

In carrying from the rejected figures, we should take what is *nearest* the truth, whether it be too great or too small.

Exam. 3. Multiply 7·24651 by 81·4632, so that there may be only three places of decimals in the product.

Here 1, the unit figure of the multiplier, is written below 6, the third decimal figure of the multiplicand; 8, the figure which *precedes* 1, is put *after* it; 4, the figure which *follows* it, is set *before* it, &c. We then say, 8 times 5 are 40, and 1 (carried for 8 times 1) are 41: 1 is then set down and 4 carried, and the rest of the work by 8 proceeds in the usual way. Then in multiplying 7·246 by 1, we add 1 to the product for 51, because 51 is nearer 100 than 0, and therefore it is nearer the truth to carry 1 than 0. In multiplying 7·24 by 4, 3 is carried for the product that would have resulted from the rejected figures: for, going two places back, we have 4 times 5 are 20; 4 times 6 are 24, and 2 are 26, which being nearer 30 than 20, we carry 3. So likewise in multiplying 7·2 by 6, we carry 3 from the rejected figures; and thus we proceed in similar cases. In finding what is to be carried for the rejected figures, it is generally sufficient to go one figure back, but in doubtful cases it may be well to go farther.

```
  7·246 51
 23 641 8
 ---------
 5 79721 −
    7247 −
    2899 −
     435 +
      22 −
       1 +
 ---------
 590·325, product.
```

The *reason* of the preceding operation will be seen from the adjoining work at full length, in which a vertical line is drawn, cutting off the part rejected in the abbreviated process. The result in this way is 590·323, or rather 590·324, on account of the following figures, and is less, by a thousandth part of a unit, than the result before obtained. The cause of this difference is, that all the partial products in the contracted mode, except the last, happen to be rather too great. If, as in the preceding example, the results which are too great be marked by the sign −, and those that are too small by +, it may enable us, in some degree, to judge of the accuracy of the result, as we may suppose it to be nearly correct, if the number of signs of each kind be

```
       7·24651
       81·4632
 --------------
       1|449302
      21|73953
     434|7906
    2898|604
    7246|51
  579720|8
 --------------
 590·323|893432
```

nearly the same, since the excesses and the defects will then probably balance each other. As the last figure, however, cannot be depended on, it is proper to work for one figure more than it is necessary to have true, and to reject it at the conclusion: and in lengthened computations, such as many of those in compound interest and annuities, it may be right to work for two or three additional figures.

Exam. 4. Multiply ·681472 by ·01286, so that the decimal in the product may contain five figures.

In this example, since the multiplier contains no integer, a cipher is placed below the fifth figure of the multiplicand, and then, the multiplier being written in reversed order, the work proceeds as in the last example.

```
·681472
 682100
 ------
    681+
    136+
     55−
      4+
 ------
 ·00876, product.
```

Exam. 5. Multiply $7·9\dot{4}$ by 3·69, so that there may be four places of decimals in the product.

Here the multiplicand is carried out to four places; and by a process similar to those which precede, the answer is found to be 29·315, which is quite correct.

```
 7·9444
    963
 ------
 238333
  47666
   7150
 ------
29·3150, product.
```

	Exercises.	*Answers.*
13.	1·123674 × 1·123674 to 6 places	1·262643
14.	$7·\dot{2}85714\dot{}$ × 36·74405 to 6 places	267·706650
15.	$24·\dot{6}\dot{3}$ × ·2347 to 6 places	$5·7821\dot{5}\dot{4}$
16.	·863541 × ·10983 to 5 places	·09484
17.	·1347866 × ·288793 to 7 places	·0389254
18.	·26736 × ·28758 to 4 places	·0769
19.	2·656419 × $1·7\dot{2}3$ to 6 places	4·578932
20.	$1·\dot{6}\dot{5}$ × $1·\dot{4}\dot{8}$ to 5 places	2·45975
21.	·053497 × ·047126 to 6 places	·002521

Exer. 22, 23, 24. Required the product, true to six places of decimals, of the numbers given in Exercises 4, 6, and 14, in the article on multiplication of fractions; the several fractions being previously reduced to decimals. *Answ.* ·113445, ·467480, and 2·393162.

25. The sun's diameter is 111·454 times the equatorial diameter of the earth, which is 7925·648 miles. Required the sun's diameter in miles. *Answ.* 883,345 miles.

26. The moon's mean distance from the earth is 29·982175 times the equatorial diameter of the earth. Required that distance in miles. (See the last exercise.) *Answ.* 237628·165 miles.

27. In consequence of the vast mass of matter contained in the

sun, a body at his surface weighs 27·9 times as much as it would at the surface of the earth. How much, then, would a person who weighs here 161 lbs. (1·4375 cwt.) weigh at the sun's surface? *Answ.* 2 tons 0 cwt. 0 qr. 11·9 lbs.

DIVISION IN DECIMAL FRACTIONS.

RULE I. (1.) If the divisor and dividend do not contain the same number of places of decimals, supply the deficiency by annexing ciphers, or, in a periodical decimal, the next figures of the period. (2.) Then, rejecting the separating points, divide as in whole numbers, and the quotient will be a whole number. (3.) If there be a remainder, after all the figures of the dividend have been used, ciphers or periodical figures may be annexed, till nothing remains, or till as many figures are found as may be judged necessary. The part of the quotient thus obtained, will be a decimal.

If, after the rejection of the separating points, the divisor be greater than the dividend, the quotient will contain no whole number, and the work will proceed according to Rule I. in reduction of decimals.

Exam. 1. Divide 1346·5 by 43·68.

Here, by annexing a cipher to the dividend, and rejecting the points, we have for divisor 4368, and for dividend 134650. Hence, dividing in the common way, we find 30 for the integral part, and annexing ciphers to the remainders, and continuing the operation, we get ·826465, &c. The answer, therefore, is 30·826465, &c. The work is left for the learner to perform.

With respect to the *reason* of the operation, the value of 1346·5 is not changed by the annexing of a cipher; and the removal of the points merely multiplies each of the given numbers by 100. (See page 181.) It is evident, therefore, that the value of the quotient will not be affected; since, while the dividend is multiplied by 100, the divisor is increased in the same ratio. We might also consider the dividend as the numerator, and the divisor as the denominator of a fraction; and then the reason of the process would depend on Proposition 1, established in page 50. The reason of removing the points is to make the dividend and divisor whole numbers, and thus to render the operation, as much as possible, the same as in simple division.

Exam. 2. Divide ·1342 by 67·1.

Here, by annexing three ciphers to the divisor, and rejecting the points, we get for divisor 671000, and for dividend 1342. Then, the divisor being the greater, the quotient will contain no integral part; and the annexing of a cipher to the dividend gives one cipher for the quotient: the annexing of a second gives another cipher; but the addition of a third gives 2. Hence, the quotient is ·002.

When the number of places of decimals in the divisor is not greater than in the dividend, the number of figures of decimals in the quotient will be equal to the difference between the number of places in the divisor and dividend, as is evident from multiplication of decimals; and in this way the number of decimal figures in the quotient is often easily determined.

Rule II. When the divisor consists of many figures, the work will be shortened, if, instead of annexing a cipher, or a periodical figure, to each remainder, a figure be cut off from the divisor. In this case, each product is to be increased by *carrying* from the product of the figure last cut off, and of the figure last placed in the quotient.

It may facilitate the use of this important contraction, if, after the rejection of the separating points, so many figures be annexed to the divisor and dividend, or taken from them, that the divisor may contain one or two figures more than are required to be in the quotient. Other directions might be given, but the following examples and illustrations will perhaps be preferable.

Exam. 3. Divide 2·3748 by 1·4736, so that the quotient may contain three places of decimals.

In this example, the numbers being prepared according to Rule I., and the first figure of the quotient being found, instead of adding a cipher to the remainder, 9012, we omit the last figure of the divisor, to denote which a point is placed below it. Then, 6 being put in the quotient, we multiply 6, the figure cut off, by it, and, without setting anything down, we carry 4, because the product 36 is nearer 40 than 30. After that, 3 is cut off in like manner, and then 7. The quotient is found to be 1·611, or more nearly 1·612, because the remainder 8 is rather more than the half of 14. The annexed operation at full length will explain the *reason* of the contracted process, the vertical line cutting off the rejected part.

```
14736)23748(1·611        14736)23748(1·611
 ···  14736                    14736
       9012                     9012|0
       8842                     8841|6
        170                      170|40
        147                      147|36
         23                       23|040
         15                       14|736
          8                        8|304
```

Exam. 4. Divide 73·64 by $\cdot 4\dot{3}\dot{2}$, so that the quotient may have four places of decimals.

Here, it is easy to see, that the quotient will contain three places of whole numbers. (This would be seen by dividing 736 by 4.) Hence the quotient must contain seven figures. Extending, therefore, the divisor to eight places, and the dividend to the same number of places of decimals, the process will stand as in the margin. In the work, instead of bringing down the last two ciphers of the dividend, two figures may be cut in succession from the divisor, and the rest of the operation will proceed as before.

```
43232323)7364000000(170·3355
  ······  43232323
          30407677
          30262626
            145051
            129697
             15354
             12970
              2384
              2162
               222
               216
                 6
```

The pupil should work all the following exercises by Rule II., and several of them by Rule I. In those in which the divisor consists of few figures, the work must be commenced by Rule I., and it may be finished by Rule II.

Exercises.	*Answers.*
1. 47·58 ÷ 26·175	1·81776504
2. ·3412 ÷ 8·4736	·040266239
3. $468{\cdot}\dot{7} \div 3{\cdot}365$	139·309889
4. $\cdot\dot{5}\dot{8} \div 77{\cdot}482$	·00756122
5. 75·347 ÷ ·3829	196·779838
6. $6{\cdot}\dot{5} \div 7{\cdot}06249$	·928222
7. 1 ÷ 10·473654	·09547766
8. 7·5 ÷ 37·38	·20064205
9. $5{\cdot}\dot{0}\dot{9} \div 6{\cdot}\dot{2}$	$\cdot\dot{8}\dot{1}$
10. $\cdot 625 \div \dot{4}2857\dot{1}$	$1{\cdot}458\dot{3}$
11. $\cdot\dot{0}\dot{9} \div \dot{2}3076\dot{9}$	$\cdot\dot{3}\dot{9}$
12. $2{\cdot}\dot{1}6\dot{6} \div 3{\cdot}125$	$\cdot 6\dot{9}1\dot{8}$
13. ·079085 ÷ ·83497	·094716
14. $\cdot\dot{6}\dot{1} \div 13{\cdot}543516$	·04549495
15. 23·6 ÷ ·037538	628·6662545
16. 7·126491 ÷ 531	13·420887005
17. ·879454 ÷ 897	·98043924
18. $52{\cdot}\dot{7}\dot{3} \div 52{\cdot}734567$	1·000053224
19. $2{\cdot}\dot{3}7\dot{0} \div 4{\cdot}\dot{9}2307\dot{6}$	$\cdot\dot{4}81$

Exer. 20. The sun contains 354,936 times as much matter as the earth, and 1048·69 times as much as Jupiter. From these data, find

how many times as much matter Jupiter contains as the earth. *Answ.* 338·456 times.

21. How many times as much matter does the earth contain as the moon, the matter in the moon being represented by 0·0125172, when that in the earth is denoted by 1? *Answ.* 79·89 times.

After the full illustration of the multiplication and division of decimals, which has been given in the preceding pages, it appears unnecessary to give their application in the rule of proportion; as, in thus applying them, the pupil can feel no difficulty, the terms being arranged in the manner already explained, and the product of the second and third terms, in like manner, divided by the first.

PRACTICE.

An ALIQUOT PART of a quantity is such a part as, when taken a certain number of times, will exactly make that quantity. Thus, 5 is an aliquot part of 20, 3 of 12, &c.

What is generally called PRACTICE in mercantile arithmetic, is only an abridged method of performing operations in the rule of proportion by means of aliquot parts; and it is chiefly employed in computing the prices of commodities.

TABLES OF ALIQUOT PARTS.

MONEY.

10*s*. 0*d*. = £$\frac{1}{2}$	2*s*. 0*d*. = £$\frac{1}{10}$	4*d*. = $\frac{1}{3}$ of 1*s*.
6*s*. 8*d*. = £$\frac{1}{3}$	1*s*. 8*d*. = £$\frac{1}{12}$	3*d*. = $\frac{1}{4}$ of 1*s*.
5*s*. 0*d*. = £$\frac{1}{4}$	1*s*. 4*d*. = £$\frac{1}{15}$	2*d*. = $\frac{1}{6}$ of 1*s*.
4*s*. 0*d*. = £$\frac{1}{5}$	1*s*. 3*d*. = £$\frac{1}{16}$	1$\frac{1}{2}$*d*. = $\frac{1}{8}$ of 1*s*.
3*s*. 4*d*. = £$\frac{1}{6}$	1*s*. 0*d*. = £$\frac{1}{20}$	1*d*. = $\frac{1}{12}$ of 1*s*.
2*s*. 6*d*. = £$\frac{1}{8}$	6*d*. = $\frac{1}{2}$ of 1*s*.	$\frac{3}{4}$*d*. = $\frac{1}{16}$ of 1*s*.

WEIGHT.

2 qrs. = $\frac{1}{2}$ cwt.	14 lbs. = $\frac{1}{8}$ cwt.	4 lbs. = $\frac{1}{28}$ cwt.
1 qr. = $\frac{1}{4}$ cwt.	8 lbs. = $\frac{1}{14}$ cwt.	14 lbs. = $\frac{1}{2}$ of a qr.
16 lbs. = $\frac{1}{7}$ cwt.	7 lbs. = $\frac{1}{16}$ cwt.	7 lbs. = $\frac{1}{4}$ of a qr.

LAND MEASURE.

2 roods = $\frac{1}{2}$ acre.	20 perches = $\frac{1}{8}$ acre.	10 perches = $\frac{1}{4}$ rood.
1 rood = $\frac{1}{4}$ acre.	16 perches = $\frac{1}{10}$ acre.	8 perches = $\frac{1}{5}$ rood.

These tables may be constructed by dividing £1, 1 acre, &c., by 2, 3, 4, &c., and selecting such of the quotients as are free from fractions. The following continuation of them will often be found useful. By its means, the pupil will be assisted in discovering what aliquot parts may, in many cases, be most advantageously employed. The more obvious and less useful parts are omitted.

MONEY.

2s. 6d. =	$\frac{1}{4}$ of 10s. 0d.
	$\frac{1}{2}$ of 5s. 0d.
1s. 8d. =	$\frac{1}{6}$ of 10s. 0d.
	$\frac{1}{3}$ of 5s. 0d.
1s. 4d. =	$\frac{1}{3}$ of 4s. 0d.
1s. 3d. =	$\frac{1}{8}$ of 10s. 0d.
	$\frac{1}{4}$ of 5s. 0d.
	$\frac{1}{2}$ of 2s. 6d.
10d. =	$\frac{1}{12}$ of 10s. 0d.
	$\frac{1}{8}$ of 6s. 8d.
	$\frac{1}{6}$ of 5s 0d.
	$\frac{1}{4}$ of 3s. 4d.
	$\frac{1}{3}$ of 2s. 6d.

10d. =	$\frac{1}{2}$ of 1s. 8d.
8d. =	$\frac{1}{6}$ of 4s. 0d.
	$\frac{1}{5}$ of 3s. 4d.
	$\frac{1}{3}$ of 2s. 0d.
7½d. =	$\frac{1}{8}$ of 5s. 0d.
	$\frac{1}{4}$ of 2s. 6d.
	$\frac{1}{2}$ of 1s. 3d.
5d. =	$\frac{1}{12}$ of 5s. 0d.
	$\frac{1}{8}$ of 3s. 4d.
	$\frac{1}{6}$ of 2s. 6d.
	$\frac{1}{4}$ of 1s. 8d.
4d. =	$\frac{1}{12}$ of 4s. 0d.

4d. =	$\frac{1}{10}$ of 3s. 4d.
	$\frac{1}{6}$ of 2s. 0d.
3d. =	$\frac{1}{10}$ of 2s. 6d.
	$\frac{1}{8}$ of 2s. 0d.
2½d. =	$\frac{1}{12}$ of 2s. 6d.
	$\frac{1}{8}$ of 1s. 8d.
	$\frac{1}{6}$ of 1s. 3d.
	$\frac{1}{4}$ of 0s. 10d.
1½d. =	$\frac{1}{10}$ of 1s. 3d.
	$\frac{1}{8}$ of 1s. 0d.
	$\frac{1}{4}$ of 0s. 6d.
¾d. =	$\frac{1}{8}$ of 0s. 6d.
	$\frac{1}{4}$ of 0s. 3d.

WEIGHT.

14 lbs. =	$\frac{1}{4}$ of 2 qrs.
8 lbs. =	$\frac{1}{7}$ of 2 qrs.
	$\frac{1}{2}$ of 16 lbs.
7 lbs. =	$\frac{1}{8}$ of 2 qrs.
	$\frac{1}{2}$ of 14 lbs.
4 lbs. =	$\frac{1}{7}$ of 1 qr.
	$\frac{1}{4}$ of 16 lbs.
2 lbs. =	$\frac{1}{8}$ of 16 lbs.
	$\frac{1}{4}$ of 8 lbs.

In the computation of prices, the quantity of the commodity may be of *one denomination* or of *more than one*; and, accordingly, the subject divides itself into two branches, with several varieties, as will appear from the following rules and illustrations.

RULE I. In finding the price of a commodity, *when the price of each article*, as well as the quantity, *is of one denomination*, multiply the price of each article by the number of articles.

Exam. 1. Required the price of 289 cwt. of beef, at £2 per cwt.

Here, the price of 289 cwt. at £1 per cwt. is evidently £289; and the price of the same, at £2 per cwt., must obviously be twice that amount.

289 cwt. at £2 per cwt.
£289 = price of 289 cwt., at £1 per cwt.
2
£578 = ——————£2———

Exercises.	*Answ.*	*Exercises.*	*Answ.*
1. 311 at £3	£933	3. 197 at £4	£788
2. 1286 at £5	£6430	4. 309 at £1	£309

RULE II. *When the price is an aliquot part of the higher denomination*; take a like part of the number of articles, and the result will be the price in the higher denomination.

Exam. 2. What cost 533 lbs. of tea, at 5*s*. per lb.?

533 lbs. at 5*s*. per lb.
£533 = price of 533 lbs. at £1 each.
5*s*. = $\frac{1}{4}$ of £1...£133 - 5 - 0 = ——— 5*s*. ——

In this example, since 533 articles, at £1 each, would cost £533, it is evident that at 5*s*. the same number of articles would cost one fourth of that amount, 5*s*. being one fourth of £1. We therefore divide £533 by 4, and the quotient, £133 - 5 - 0, is the required price.

Exam. 3. Find the price of 537 yards, at 3*d*. per yard.

537 yards at 3*d*. per yard.
537*s*. = price of 537 yards at 1*s*. 0*d*. per yard.
3*d*. = $\frac{1}{4}$ of 1*s*. 0*d*....134*s*. 3*d*. = ——— 3*d*. ———
£6 - 14 - 3, *answ*.

Here, 537 yards at 1 shilling would cost 537 shillings; and the price of the same at 3*d*. would evidently be a fourth of that, or 134*s*. 3*d*., which, by reduction, becomes £6 - 14 - 3.

Exercises.	*Answers.*	*Exercises.*	*Answers.*
5. 389 at 10*s*. 0*d*.	£194 - 10 - 0	7. 645 at 2*s*. 6*d*.	£80 - 12 - 6
6. 739 at 3*s*. 4*d*.	£123 - 3 - 4	8. 2571 at 0*s*. 2*d*.	£21 - 8 - 6

RULE III. The given price of each article may often be divided into parts, such that the values of the whole, at these rates taken separately, may be found by the first or second rule, or that the values of some of them may be derived from those at others. The sum of the results thus obtained, will be the price required.

Exam. 4. What cost 479 cwt. of sugar, at £4 - 9 - 6 per cwt.?

	479 cwt. at £4 - 9 - 6 per cwt.		
	£479	= price of 479 cwt. at £1 per cwt.	
	4		
	£1916	= ———	£4 ——
5*s*. 0*d*. = $\frac{1}{4}$ of £1	119 - 15 - 0	= ———	5*s*. ——
4*s*. 0*d*. = $\frac{1}{5}$ of £1	95 - 16 - 0	= ———	4*s*. ——
0*s*. 6*d*. = $\frac{1}{10}$ of 5*s*.	11 - 19 - 6	= ———	6*d*. ——
Answ.	£2143 - 10 - 6	= ———	£4 - 9 - 6 ——

In this example, at £1 per cwt., 479 cwt. would cost £479; and hence, to find the price at £4 per cwt., we multiply, according to Rule I., by 4. Again, since *5s.* = one fourth of a pound, and *4s.* = one fifth of a pound, the prices at these rates will be found, by Rule II., by taking a fourth and a fifth of £479. It still remains to find the price at *6d.* Now, since *6d.* is one tenth of *5s.*, the price at *6d.* will evidently be one tenth of the price at *5s.* ∴ we take, therefore, one tenth of £119 - 15, the price at *5s.*, and the result is £11 - 19 - 6, the price of 479 cwt. at *6d.* per cwt. Finally, the price of the whole quantity at £4 per cwt. being £1916; at *5s.*, £119 - 15; at *4s.*, £95 - 16; and at *6d.*, £11 - 19 - 6 the price at £4 - 9 - 6 will be £2143 - 10 - 6, the sum of all these.

Exam. 5. What cost 647 yards of linen, at *3s. 9d.* per yard?

	647 yards at *3s. 9d.* per yard.	
	£647	= price of 647 yards at £1 per yard.
2s. 6d. = $\frac{1}{8}$ of £1	80 - 17 - 6 =	*2s. 6d.*
1s. 3d. = $\frac{1}{2}$ of *2s. 6d.*	40 - 8 - 9 =	*1s. 3d.*
Answ.	£121 - 6 - 3 =	*3s. 9d.*

In this example, the price at £1 per yard is £647; and, as *2s. 6d.* is one eighth of a pound, the price at *2s. 6d.* will be one eighth of £647, or £80 - 17 - 6. Now, *2s. 6d.* and *1s. 3d.* make up the given price, *3s. 9d.*; and *1s. 3d.* being one half of *2s. 6d.*, the price at *1s. 3d.* will be half the price at *2s. 6d.*, and hence we take the half of £80 - 17 - 6. Then, the prices at *2s. 6d.* and *1s. 3d.* being £80 - 17 - 6, and £40 - 8 - 9, the price at *2s. 6d.* and *1s. 3d.*, or at *3s. 9d.*, will be £121 - 6 - 3, the sum of these. This exercise might be wrought with nearly equal facility by finding the prices at *3s. 4d.* and *5d.*, and taking their sum.

Exam. 6. Find the price of 247 cwt. of flour, at £1 - 5 per cwt.

	247 cwt. at £1 - 5 per cwt.	
	£247......	= price at £1 per cwt.
5s. 0d. = $\frac{1}{4}$ of £1...	61 - 15 =	*5s.*
Answ.	£308 - 15 =	£1 - 5

The examples, thus far, have been wrought at full length, for explaining the operations; and the pupil should thus work similar questions, till he thoroughly understand the reason of the process. Afterwards, however, the work may stand as in the margin, which is the mode commonly employed.

	247 cwt. at £1 - 5 per cwt.
5s. = £$\frac{1}{4}$	61 - 15
	£308 - 15, *answ.*

Exam. 7. What is the cost of 195 lbs. of raisins, at *1s. 3d.* per lb.?

	195 lbs. at *1s. 3d.* per lb.	
1s. 3d. = $\frac{1}{16}$ of £1... {	4)£195	= price at £1 per lb.
	4)48 - 15	
	Answer, £12 - 3 - 9 =	*1s. 3d.*

Or thus, 195 lbs. at 1*s.* 3*d.* per lb.

	195*s.*	= price at 1*s.* 0*d.* per lb.
3*d.* = $\frac{1}{4}$ of 1*s.* 0*d.*...	48 - 9 ...	= ——— 0*s.* 3*d.* ——
	2|0)243 - 9 ...	= ——— 1*s.* 3*d.* ——
	£12 - 3 - 9, *answ.*, as before.	

In the first of the preceding operations, since 1*s.* 3*d.* is one sixteenth of a pound, we divide £195, the price at a pound sterling per pound, by 4, and the result again by 4. In the second method, the price at one shilling being 195 shillings, and the price at 3*d.* one fourth of that amount, or 48*s.* 9*d.*, the sum of both is 243*s.* 9*d.*; or, by reduction, £12 - 3 - 9, the same as before. This operation may also stand as in the margin.

	195 lbs. at 1*s.* 3*d.* per lb.
3*d.* = $\frac{1}{4}$ of 1*s.* 0*d.*	48 - 9
	2|0)24|3 - 9
	£12 - 3 - 9, *answ.*

Exam. 8. What cost 1257 yards of ribbon at 6$\frac{3}{4}$*d.* per yard?

	1257 yards at 6$\frac{3}{4}$*d.* per yard.	
6*d.* = $\frac{1}{2}$ of 1*s.* 0*d.*	628 - 6	= price at 6*d.* per yard.
$\frac{3}{4}$*d.* = $\frac{1}{8}$ of 6*d.*	78 - 6$\frac{3}{4}$	= ——— $\frac{3}{4}$*d.* — —
	2|0)70|7 - 0$\frac{3}{4}$	= ——— 6$\frac{3}{4}$*d.* ———
	£35 - 7 - 0$\frac{3}{4}$, *answ.*	

Exam. 9. What is the cost of 347 cwt. of coffee, at £7 - 11 - 6 per cwt.?

	347 cwt. at £7 - 11 - 6 per cwt.	
	7	
	£2429	= price at £7 - 0 - 0 per cwt.
10*s.* = $\frac{1}{2}$ of £1	173 - 10 - 0	= ——— £0 - 10 - 0 ———
1*s.* 3*d.* = $\frac{1}{8}$ of 10*s.*	21 - 13 - 9	= ——— 0 - 1 - 3 ———
0*s.* 3*d.* = $\frac{1}{5}$ of 1*s.* 3*d.*	4 - 6 - 9	= ——— 0 - 0 - 3 ———
Answ.	£2628 - 10 - 6	= ——— £7 - 11 - 6 ———

Or thus, 347 cwt. at £7 - 11 - 6 per cwt.

	7	
	£2429	= price at £7 - 0 - 0 per cwt.
10*s.* 0*d.* = $\frac{1}{2}$ of £1	173 - 10 - 0	= ——— 0 - 10 - 0 ———
1*s.* 0*d.* = $\frac{1}{10}$ of 10*s.*	17 - 7 - 0	= ——— 0 - 1 - 0 ———
0*s.* 6*d.* = $\frac{1}{2}$ of 1*s.* 0*d.*	8 - 13 - 6	= ——— 0 - 0 - 6 ———
Answ.	£2628 - 10 - 6	= ——— £7 - 11 - 6 ———

Exam. 10. If a tradesman have 3*s.* 9$\frac{1}{2}$*d.* per day, how much is his yearly salary, the days of labour in the year being 313?

	313 days at 3*s.* 9$\frac{1}{2}$*d.* per day.	
3*s.* 4*d.* = $\frac{1}{6}$ of £1	£52 - 3 - 4	= amount at 3*s.* 4*d.* per day.
0*s.* 5*d.* = $\frac{1}{8}$ of 3*s.* 4*d.*	6 - 10 - 5	= ——— 0*s.* 5*d.* ———
0*s.* 0$\frac{1}{2}$*d.* = $\frac{1}{10}$ of 5*d.*	0 - 13 - 0$\frac{1}{2}$	= ——— 0*s.* 0$\frac{1}{2}$*d.* ———
Answ.	£59 - 6 - 9$\frac{1}{2}$	= ——— 3*s.* 9$\frac{1}{2}$*d.* ———

This exercise might have been wrought by distributing 3*s*. $9\frac{1}{2}d.$ into the parts, 2*s*. 6*d*., 1*s*. 3*d*., and $\frac{1}{2}d.$; but those employed above are much preferable, as, in the other case, $\frac{1}{2}d.$ being one thirtieth of 1*s*. 3*d*., we must have divided by a number inconveniently large. Unless, indeed, in particular circumstances, we should, if possible, avoid taking parts that would require us to divide by any number greater than 12. The seventh example affords an instance in which no inconvenience arises from employing a large divisor; and other instances will frequently occur.

In taking aliquot parts, it sometimes shortens the work to take the same part twice, as a result may thus be copied without working for it again. Thus, 18*s*. 6*d*. may be divided into 10*s*., 4*s*., 4*s*., and 6*d*. Sometimes also the price at a small rate may be found, and from it the price at a greater may be obtained by multiplication. Thus, 16*s*. 4*d*. may be divided into 2*s*., 14*s*., and 4*d*.; the price at 14*s*. being 7 times the price at 2*s*. In like manner, for 17*s*. 1*d*. we may take 1*s*. 8*d*., 15*s*. (9 times 1*s*. 8*d*.), and 5*d*. Other remarks on this subject will be found at the end of this article.

	Exercises.	£	*s.*	*d.*	*Answers.* £	*s.*	*d.*
9.	1625 at	0	2	$8\frac{1}{2}$	220	6	$0\frac{1}{2}$
10.	1429	1	2	9	1625	9	9
11.	1973	0	6	10	674	2	2
12.	749	0	5	8	212	4	4
13.	1689	0	4	$10\frac{1}{2}$	411	13	$10\frac{1}{2}$
14.	2476	0	18	6	2290	6	0
15.	5926	0	11	8	3456	16	8
16.	313	0	8	8	135	12	8
17.	5934	1	5	10	7664	15	0
18.	3576	0	11	$4\frac{1}{2}$	2033	17	0
19.	958	1	18	8	1852	2	8
20.	1898	0	6	$10\frac{1}{2}$	652	8	9
21.	1594	0	13	6	1075	19	0
22.	695	0	14	10	515	9	2
23.	2386	1	6	6	3161	9	0
24.	725	1	7	8	1002	18	4
25.	589	1	11	6	927	13	6
26.	286	0	12	1	172	15	10
27.	7649	0	0	$5\frac{3}{4}$	183	5	$1\frac{3}{4}$
28.	5728	0	0	$7\frac{1}{4}$	173	0	8
29.	6491	0	0	$10\frac{3}{4}$	290	14	$10\frac{1}{4}$
30.	991	0	1	$3\frac{1}{2}$	64	0	0
31.	436	4	17	8	2129	2	8
32.	3725	0	5	$8\frac{1}{4}$	1059	5	$11\frac{1}{4}$
33.	1677	0	5	1	426	4	9

	Exercises.	£ s. d.		*Answers.* £ s. d.
34.	7913	2 16 $10\frac{1}{2}$	....	22502 11 $10\frac{1}{2}$
35.	4265	1 14 $1\frac{1}{2}$	....	7277 3 $1\frac{1}{2}$
36.	249	5 13 9	....	1416 3 9
37.	576	0 18 9	....	540 0 0
38.	6485	2 10 6	....	16374 12 6

RULE IV. *To find the price of any number of articles at 2 shillings each:* Double the last figure of the number for shillings, and take the number expressed by the preceding figures as pounds.

Exam. 11. What is the cost of 647 yards of muslin, at 2*s.* per yard?

The *reason of this rule* is, that 2 shillings are one tenth of a pound, and the work is no more than a contracted division by ten. Thus, in the annexed example, by dividing by 10, we should have £64, with the fraction £$\frac{7}{10}$, or, by doubling both the terms, £$\frac{14}{20}$, or 14 shillings.

647 yards at 2*s.*
£64 - 14 - 0, *answ.*

RULE V. *If the rate be an even number of shillings,* multiply by half that number, and in multiplying, double the last figure of the product for shillings; the rest will be pounds.

Exam. 12. Required the price of 273 ℔s. of indigo, at 8 shillings per ℔.

The operation by this rule is evidently nothing more than an extension of the last. In the annexed example, the price is $\frac{8}{20}$ or $\frac{4}{10}$ of a pound; and hence we multiply by 4, and the doubling of the last figure of the product for shillings, is equivalent to a division by 10.

273 ℔s. at 8*s.* per ℔.
4
£109 - 4 - 0, *answ.*

RULE VI. *When the number of shillings is odd,* we may find, by the last rule, the amount at one shilling less than the given rate, and for that shilling take one twentieth of the price at one pound.

Exam. 13. What is the cost of 787 pounds of nutmegs, at 17 shillings per pound?

In this example, the price is first found at 16*s.* per pound, by multiplying by 8, and doubling the last figure of the product for shillings: then, for the remaining shilling, a twentieth part of £787, the price at 20*s.* per ℔., is taken; and the sum of both results is £668 - 19, the price at 17 shillings.

	787 ℔s. at 17*s.* per ℔.	
	8	
	£629 - 12	= price at 16*s.* per ℔.
1*s.* = £$\frac{1}{20}$	39 - 7	= ——— 1*s.* ——
	£668 - 19	= ——— 17*s.* ——

Exam. 14. Required the amount of 233 cwt. of pearl ashes, at £3 - 9 per cwt.

In this example, we first find the price at £3, by multiplying by 3; then the price at 8*s.*, by multiplying by 4, and doubling the last figure of the product for shillings; and, lastly, the price at 1*s.*, by taking a twentieth part of £233, the price at £1 per cwt.

	233 cwt. at £3 - 9 per cwt.	
	3 4	
	699	= price at £3 per ℔.
	93 - 4	= ——— 8*s.* ———
1*s.* = £$\frac{1}{20}$	11 - 13	= ——— 1*s.* ———
Answ.	£803 - 17	= ——— £3 - 9 ———

It is evident, that we might have taken an eighth of the price at eight shillings, for the price at one shilling.

Exercises.		*Answers.* £	*s.*	*d.*	*Exercises.*		*Answers.* £	*s.*	*d.*
39. 397 at	2*s.*	39	14	0	46. 675 at	13*s.* ...	438	15	0
40. 418	3*s.*	62	14	0	47. 512	14*s.* ...	358	8	0
41. 763	6*s.*	228	18	0	48. 319	16*s.* ...	255	4	0
42. 378	7*s.*	132	6	0	49. 868	17*s.* ...	737	16	0
43. 841	8*s.*	336	8	0	50. 470	18*s.* ...	423	0	0
44. 937	9*s.*	421	13	0	51. 983	£3 - 4	3145	12	0
45. 949	12*s.*	569	8	0	52. 724	£5 - 13	4090	12	0

RULE VII. The price may often be determined very easily, by finding the amount at a rate higher than the given rate, and deducting from the amount the price at the difference between the given rate and the assumed one.

This method is generally of little use, unless the difference between the given and the higher rate be an aliquot part of the higher. This difference may be called the COMPLEMENT of the given rate.

Exam. 15. What cost 189 tons of coals, at 17*s.* 6*d.* per ton?

	119 tons, at 17*s.* 6*d.* per ton.	
	£189	= price at £1 per ton.
2*s.* 6*d.* = £$\frac{1}{8}$	23 - 12 - 6	= ——— 2*s.* 6*d.* ———
Answ.	£165 - 7 - 6	= ——— 17*s.* 6*d.* ———

Or simply thus:

	189 at 17*s.* 6*d.*
2*s.* 6*d.* = £$\frac{1}{8}$ ...	23 - 12 - 6
	£125 - 7 - 6

In this example, since 17*s.* 6*d.* is less than a pound by 2*s.* 6*d.*, which is an aliquot part of a pound, the required price is found by taking from £189, the price at £1 per ton, £23 - 12 - 6, the price at 2*s.* 6*d.* per ton.

Exam. 16. What is the cost of 257 feet of plank, at 10*d.* per foot?

Here, from 257*s.*, the price at 1*s.* per foot, we take 42*s.* 10*d.*, the price at 2*d.* per foot: the remainder is 214*s.* 2*d.*, or £10 - 14 - 2, the price at 10*d.* per foot.

$$\begin{array}{rl} & 257 \text{ feet at } 10d. \text{ per foot.} \\ 2d. = \frac{1}{6} \text{ of } 1s. \ldots & \underline{42\ \ 10} \\ & 2|0)\overline{21|4 - 2} \\ & \overline{£10 - 14 - 2,}\ \textit{answ.} \end{array}$$

Exam. 17. What is the cost of 514 gallons of seal oil, at 3*s.* 11*d.* per gallon?

$$\begin{array}{rlll} & 514 \text{ gallons, at } 3s.\ 11d. \text{ per gallon.} & & \\ & \underline{\quad 4} & & \\ & 2056 \text{ shillings} = & \text{price at } 4s. & \text{per gallon.} \\ 1d. = \frac{1}{12} \text{ of } 1s. \ldots & \underline{\ \ 42 - 10} \quad = & \text{———} \ \ \underline{1d.} & \text{———} \\ & 2|0)\overline{201|3 - \ 2} \quad = & \text{———} \ \ 3s.\ 11d. & \text{———} \\ & \overline{£100 - 13 - 2,}\ \textit{answ.} & & \end{array}$$

Exam. 18. Required the price of 193 cwt. of Turkey figs, at £3 - 18 per cwt.

In this example, the price at 2*s.* per cwt. (found by Rule IV.) is taken from the price at £4.

$$\begin{array}{lll} 193 \text{ cwt. at } £3 - 18 \text{ per cwt.} & & \\ \underline{\quad 4} & & \\ £772 \ldots\ldots = \text{price at} & £4 - \ 0 & \text{per cwt.} \\ \underline{\ \ 19 - \ 6} = \text{———} & \underline{\ \ 0 - \ 2} & \text{———} \\ £752 - 14 = \text{———} & £3 - 18 & \text{———} \end{array}$$

Exercises.		£	*s.*	*d.*		*Answers.* £	*s.*	*d.*
53.	358 at	0	13	4	....	238	13	4
54.	599	0	0	10½	....	26	4	1½
55.	967	0	0	8	....	32	4	8
56.	275	2	15	0	....	756	5	0
57.	361	3	12	0*	....	1299	12	0
58.	889	0	0	9	....	33	6	9
59.	483	5	16	8	....	2817	10	0
60.	189	0	4	7	....	43	6	3
61.	997	0	0	11	....	45	13	11
62.	753	0	1	9	....	65	17	9
63.	649	0	1	5½	....	47	6	5½
64.	721	0	19	0	....	684	19	0

We now proceed to problems of the second class; and in resolving them, we may employ either of the two following general rules.

* £3 - 12 is the difference between £4 and 8 shillings, and for 8*s.* we are to take a tenth of the price at £4. Exercise 62 will be wrought by finding the price at two shillings, and then proceeding in the usual manner; and the answers of Exercises 60 and 63 may be derived from the prices at 5*s.* 0*d.* and 1*s.* 8*d.*

RULE VIII. *When the quantity is not expressed by a whole number of one denomination*: (1.) Compute the price of the integral part by some of the methods already given: (2.) Then find the price of the fractional parts, or lower denominations, from the given rate, by means of aliquot parts, or otherwise: (3.) The sum of all will be the whole price required. Or,

RULE IX. *When the quantity is not expressed by a whole number of one denomination*: (1.) Find the price of the entire given quantity at £1 * for each unit of the integral part, valuing the subordinate parts at the same rate: (2.) Then the operation will proceed in the manner already explained, without farther work for the subordinate parts.

Exam. 19. Required the price of $79\frac{3}{4}$ yards of broadcloth, at £1 - 2 - 11 per yard.

By Rule VIII.

	$79\frac{3}{4}$ yards, at £1 - 2 - 11.
2s. 6d. = £$\frac{1}{8}$	9 - 17 - 6 = price of 79 yards, at *2s. 6d.*
5d. = $\frac{1}{6}$ of *2s. 6d.*	1 - 12 - 11 = ——— *5d.*
$\frac{1}{2}$ of £1 - 2 - 11	0 - 11 - $5\frac{1}{2}$ = ——— $\frac{1}{2}$ yard
$\frac{1}{2}$ of *11s. $5\frac{1}{2}$d.*	0 - 5 - $8\frac{3}{4}$ = ——— $\frac{1}{4}$ ———
	£91 - 7 - $7\frac{1}{4}$, *answ.*

In this process, the price of 79 yards is first found (or rather the parts composing it are found); and then, for $\frac{1}{2}$ yard, the half of £1 - 2 - 11 is taken, and for $\frac{1}{4}$ yard, the half of that is found: the sum of all which parts is £91 - 7 - $7\frac{1}{4}$, the result required.

By Rule IX.

	$79\frac{3}{4}$ yards, at £1 - 2 - 11.
	£79 - 15 - 0 = price at £1......... per yard.
2s. 6d. = £$\frac{1}{8}$	9 - 19 - $4\frac{1}{2}$ = ——— *2s. 6d.* ———
0s. 5d. = $\frac{1}{6}$ of *2s. 6d.*	1 - 13 - $2\frac{3}{4}$ = ——— *5d.* ———
Answ.	£91 - 7 - $7\frac{1}{4}$ = ——— £1 - 2 - 11 ———

* This rule is restricted to *pounds*, as a farther extension of it is of little use. The principle, however, is universal in its nature, and may be extended to any other rate in a similar manner.

With respect to the eighth and ninth rules, it may be observed, that the ninth (of which Rules X., XI., XII., and XIII. are particular applications) is very elegant in principle; and, in many cases, it affords the neatest and most concise mode of computing prices. In some cases, however, it gives origin to fractions, which, when perfect accuracy is required, render the operation tedious. In such cases, the eighth rule is sometimes preferable. For mercantile purposes, however, as it is unnecessary to work for the precise fractions, the ninth rule, when managed as will be shown in the succeeding rules, will perhaps be found superior in all the principal and more difficult computations.

In this method, we first find the price of $79\frac{3}{4}$ yards, at £1 per yard. This is £79 - 15; for the price of 79 yards is £79, and, the price of a quarter of a yard being evidently 5*s.* 0*d.*, that of $\frac{3}{4}$ of a yard is 15*s.* Then, the price at £1 per yard being £79 - 15, the price at 2*s.* 6*d.* will be one eighth of this, or £9 - 19 - $4\frac{1}{2}$; and the price at 5*d.*, one sixth of the price at 2*s.* 6*d.*, or £1 - 13 - $2\frac{3}{4}$. The sum of these is £91 - 7 - $7\frac{1}{4}$, the whole price, as before.

Exam. 20. What is the cost of $69\frac{5}{8}$ yards of cambric, at 13*s.* 10*d.* per yard?

By Rule VIII.

Here, after finding as already illustrated, the parts whose sum will be the price of 69 yards, at 13*s.* 10*d.* per yard, we take, for $\frac{4}{8}$ of a yard, the half of 13*s.* 10*d.*; and taking $\frac{1}{4}$ of 6*s.* 11*d.*, we get 1*s.* $8\frac{3}{4}$*d.*, the price of $\frac{1}{8}$ of a yard. The sum of all these partial prices is £48 - 3 - $1\frac{3}{4}$, the whole price required.

	$69\frac{5}{8}$ at 13*s.* 10*d.*
10*s.* 0*d.* = £$\frac{1}{2}$	£34 - 10 - 0
3*s.* 4*d.* = $\frac{1}{3}$ of 10*s.*	11 - 10 - 0
0*s.* 6*d.* = $\frac{1}{20}$ of 10*s.*	1 - 14 - 6
$\frac{4}{8}$ yard = $\frac{1}{2}$ yard	0 - 6 - 11
$\frac{1}{8}$ yard = $\frac{1}{4}$ of $\frac{4}{8}$ yard	0 - 1 - $8\frac{3}{4}$
Answ.	£48 - 3 - $1\frac{3}{4}$

By Rule IX.

	$69\frac{5}{8}$ yards, at 13*s.* 10*d.* per yard.	
	£69 - 12 - 6 = price at £1	.........per yard.
10*s.* 0*d.* = £$\frac{1}{2}$	34 - 16 - 3 = ——— 10*s.*	———
3*s.* 4*d.* = $\frac{1}{3}$ of 10*s.*	11 - 12 - 1 = ——— 3*s.* 4*d.*	———
0*s.* 6*d.* = $\frac{1}{20}$ of 10*s.*	1 - 14 - $9\frac{3}{4}$ = ——— 0*s.* 6*d.*	———
Answ.	£48 - 3 - $1\frac{3}{4}$ = ——— 13*s.* 10*d.*	———

In this method, since, at £1 per yard, one eighth of a yard would cost 2*s.* 6*d.*, five eighths would cost 5 times 2*s.* 6*d.*, or 12*s.* 6*d.*; and, consequently, $69\frac{5}{8}$ yards would cost £69 - 12 - 6. The rest of the work proceeds in the usual manner. With respect to both modes, it may be observed, that, on account of the large divisor, 20, the work would have perhaps been as short, had we taken 10*s.*, 3*s.* 4*d.*, 5*d.*, and 1*d.*, though it would require an additional line.

Exercises.			£	*s.*	*d.*		*Answers.* £	*s.*	*d.*
65.	$328\frac{3}{4}$	at	0	6	6	....	106	16	$10\frac{1}{2}$
66.	$675\frac{1}{2}$		2	14	4	....	1835	2	2
67.	$7538\frac{3}{4}$		0	2	4	....	879	10	5
68.	176c. 3q.		0	18	8	....	164	19	4
69.	$164\frac{7}{8}$		2	5	6	....	375	1	$9\frac{3}{4}$
70.	239a. 3r.		0	10	10	....	129	17	$3\frac{1}{2}$
71.	$172\frac{7}{10}$		3	15	10	....	654	16	5
72.	$257\frac{11}{12}$		2	9	4	....	636	3	$10\frac{1}{2}$

RULE X. *In calculating the prices of hundreds, quarters, and pounds*, the price, at £1 per cwt., will be found by multiplying the pounds by $2\frac{1}{7}$, and considering the product as pence; and by multiplying the quarters by 5, and considering the product as shillings.

The *reason of this rule* is evident; since, at £1 per cwt., each quarter would evidently cost 5*s.*, and each pound a twenty-eighth of this, or $2\frac{1}{7}$ pence.

Exam. 21. Required the price of 198 cwt. 2 qrs. 21 ℔s. of cast steel, at £4 - 8 - 8 per cwt.

By Rule VIII.

	c. q. ℔s.	
	198 2 21 at £4 - 8 - 8 per cwt.	
	4	
	£792 - 0 - 0	= price of 198 cwt., at £4 per cwt.
8*s.* 0*d.* = $\frac{1}{10}$ of £4	79 - 4 - 0	= ——————— 8*s.* ———
0*s.* 8*d.* = $\frac{1}{12}$ of 8*s.*	6 - 12 - 0	= ——————— 8*d.* ———
2 qrs. = $\frac{1}{2}$ of 1 cwt.	2 - 4 - 4	= price of 2 qrs. at £4 - 8 - 8 per cwt.
4 ℔s. = $\frac{1}{2}$ of 2 qrs.	0 - 11 - 1	= ——— 14 ℔s. ———————
7 ℔s. = $\frac{1}{2}$ of 14 ℔s.	0 - 5 - $6\frac{1}{2}$	= ——— 7 ℔s. ———————
	£880 - 16 - $11\frac{1}{2}$	= price required.

By Rule X.

	c. q. ℔s.	
	198 2 21 at £4 - 8 - 8 per cwt.	
	5 $2\frac{1}{7}$	
	£198 - 13 - 9	= price of 198 c. 2 q. 21 ℔s., at £1 - 0 - 0 per cwt.
	4	
	794 - 15 - 0	= ——————————— 4 - 0 - 0 ———
$\frac{1}{10}$	79 - 9 - 6	= ——————————— 0 - 8 - 0 ———
$\frac{1}{12}$	6 - 12 - $5\frac{1}{2}$	= ——————————— 0 - 0 - 8 ———
An.	880 - 16 - $11\frac{1}{2}$	= ——————————— £4 - 8 - 8 ———

The first mode of working this exercise is sufficiently explained in the operation itself. In the second mode, we say, one seventh of 21 is 3, and twice 21 are 42, and 3 are 45 pence, or 3*s.* 9*d.*, which is the value of the pounds at £1 per cwt. We then set down 9*d.* and carry 3 to 5 times 2, or 10 shillings, the price of 2 quarters; and we thus find the price of 2 q. 21 ℔s., at £1 per cwt., to be 13*s.* 9*d.*; to which £198, the price of 198 cwt. at the same rate, is prefixed. The rest of the operation proceeds in the usual manner.

Exercises. *Answers.*

	cwt.	q.	℔s.		£	*s.*	*d.*		£	*s.*	*d.*
73.	75	3	21	at	4	14	6	per cwt. ...	358	16	1
74.	285	3	7		3	17	8	——— ...	1109	18	$1\frac{1}{4}$
75.	117	1	7		0	17	4	——— ...	101	13	5

	Exercises.						*Answers.*		
	cwt.	q.	lbs.	£	s.	d.	£	s.	d.
76.	84	3	14	12	11	8 per cwt. ...	1068	0	$2\frac{1}{2}$
77.	134	1	21	0	18	4 ——— ...	123	4	$8\frac{1}{4}$
78.	836	2	21	2	8	4 ——— ...	2021	19	$10\frac{3}{4}$
79.	812	3	7	6	12	8 ——— ...	5391	13	$1\frac{1}{2}$
80.	176	1	14	2	16	10 ——— ...	501	3	$11\frac{3}{4}$

Exam. 22. Required the price of 319 cwt. 3 qrs. 16 lbs. of glue, at £2 - 12 - 6 per cwt.

By Rule VIII.

	q. lbs.	
	319 3 16 at £2 - 12 - 6 per cwt.	
	2	
	£638 - 0 - 0	= price of 319 cwt. at £2 per cwt.
10s. = £$\frac{1}{2}$	159 - 10 - 0	= ——————— 10s. ———
2s. 6d. = $\frac{1}{4}$ of 10s.	39 - 17 - 6	= ——————— 2s. 6d. ———
2 qrs. = $\frac{1}{2}$ of 1 c.	1 - 6 - 3	= price of 2 q. at £2 - 12 - 6 per cwt.
1 qr. = $\frac{1}{2}$ of 2 q.	0 - 13 - $1\frac{1}{2}$	= ——— 1qr. ———————
16 lbs. = $\frac{1}{7}$ of 1 c.	0 - 7 - 6	= ——— 16 lbs. ———————
Answ.	£839 - 14 - $4\frac{1}{2}$, the price required.	

By Rule X.

	c. q. lbs.	
	319 3 16 at £2 - 12 - 6 per cwt.	
	5 $2\frac{1}{7}$	
	£319 - 17 - $10\frac{2}{7}$	= price at £1 - 0 - 0 per cwt.
	2	
	£639 - 15 - $8\frac{4}{7}$	= ——— £2 - 0 - 0 ———
10s. 0d. = £$\frac{1}{2}$	159 - 18 - $11\frac{1}{7}$	= ——— 0 - 10 - 0 ———
2s. 6d. = $\frac{1}{4}$ of 10s.	39 - 19 - $8\frac{11}{14}$	= ——— 0 - 2 - 6 ———
Answ.	£839 - 14 - $4\frac{1}{2}$	= ——— £2 - 12 - 6 ———

The latter of these methods gives origin to fractions, which render the operation more tedious. This may be obviated by proceeding on the same principle, but converting the fractions into decimals, which it will be found to be sufficient to carry out to two places each. Thus, the work may be as follows:—

	cwt. q. lbs.
	319 3 16 at £2 - 12 - 6 per cwt.
	5 $2\frac{1}{7}$
	£319-17-10·29
	2
	639-15- 8·58
10s. 0d. = £$\frac{1}{2}$	159-18-11·14
2s. 6d. = $\frac{1}{4}$ of 10s.	39-19- 8·78
Answ.	£839-14- 4·50, or £839 - 14 - $4\frac{1}{2}$, as before.

Here, in taking $\frac{1}{7}$ of 16, we have 2 to carry, and 2 remaining; then, conceiving a cipher annexed to the remainder, and dividing 20 by 7, we set down the quotient 2, and conceiving a cipher annexed to the remainder 6, we have 7 contained most nearly 9 times in 60. We then proceed as before, and find for the price at £1 per cwt. £319 - 17 - 10·29 nearly. After this the work proceeds as before, only that in each line the pence and the decimal are multiplied and divided as if they were a single whole number, the point being retained. Thus, in finding the price at 2*s.* 6*d.*, after having found £39 - 19, we have 2*s.* 11*d.*, or 35*d.*, remaining; we then divide 35·14 by 4, as if it were all one number, and find for the quotient 878, or, with the point, 8·78. The final result is £839 - 14 - 4·50, or £839 - 14 - 4½, the same as by the other processes.

In working by this method, it should be recollected that ·25*d.* *is a farthing; ·50d. a halfpenny; and ·75d. three farthings; and, in valuing the decimal found in the answer, the pupil should consider to which of these it is nearest, and value it accordingly.*

It may be observed, that, as the method of managing the decimals in all operations of this kind is uniformly the same, *any pupil may practise this method, whether he has studied decimal fractions or not.*

The following question is wrought in several ways, for the purpose of showing their comparative advantages, and of affording an additional example of the mode of performing computations of this kind.

Exam. 23. Required the price of 212 cwt. 3 qrs. 19 lbs. of barilla, at £1 - 13 - 2 per cwt.

By Rule VIII.

	cwt. q. lbs.
	212 3 19 at £1 - 13 - 2 per cwt.
10*s.* 0*d.* = £$\frac{1}{2}$	106 - 0 - 0
2*s.* 0*d.* = £$\frac{1}{10}$	21 - 4 - 0
1*s.* 0*d.* = $\frac{1}{2}$ of 2*s.*	10 - 12 - 0
0*s.* 2*d.* = $\frac{1}{6}$ of 1*s.*	1 - 15 - 4
2 qrs. = $\frac{1}{2}$ of 1 cwt. ...	0 - 16 - 7
1 qr. = $\frac{1}{2}$ of 2 qrs. ...	0 - 8 - $3\frac{1}{2}$
16 lbs. = $\frac{1}{7}$ of 1 cwt. ...	0 - 4 - $8\frac{6}{7}$
2 lbs. = $\frac{1}{8}$ of 16 lbs....	0 - 0 - $7\frac{3}{28}$
1 lb. = $\frac{1}{2}$ of 2 lbs.	0 - 0 - $3\frac{31}{56}$
Answ.	£353 - 1 - $10\frac{1}{56}$

Or thus:

	cwt. q. lbs.
	212 3 19 at £1 - 13 - 2 per cwt.
10*s*. 0*d*. = £$\frac{1}{2}$	106 - 0 - 0
2*s*. 0*d*. = £$\frac{1}{10}$	21 - 4 - 0
1*s*. 0*d*. = $\frac{1}{2}$ of 2*s*.	10 - 12 - 0
0*s*. 2*d*. = $\frac{1}{6}$ of 1*s*.	1 - 15 - 4
2 qrs. = $\frac{1}{2}$ of 1 cwt. ...	0 - 16 - 7
1 qr. = $\frac{1}{2}$ of 2 qrs. ...	0 - 8 - 3·50
16 lbs. = $\frac{1}{7}$ of 1 cwt. ...	0 - 4 - 8·86
2 lbs. = $\frac{1}{8}$ of 16 lbs....	0 - 0 - 7·11
1 lb. = $\frac{1}{2}$ of 2 lbs. ...	0 - 0 - 3·55
Answ.	£353 - 1 - 10·02, or
	£353 - 1 - 10, nearly.

By Rule X.

	cwt. q. lbs.
	212 3 19
	5 $2\frac{1}{7}$
	£212 - 18 - $4\frac{5}{7}$
10*s*. = £$\frac{1}{2}$	106 - 9 - $2\frac{5}{14}$
2*s*. = £$\frac{1}{10}$	21 - 5 - $10\frac{1}{14}$
1*s*. = $\frac{1}{2}$ of 2*s*.	10 - 12 - $11\frac{1}{28}$
2*d*. = $\frac{1}{6}$ of 1*s*.	1 - 15 - $5\frac{47}{56}$
Answ.	£353 - 1 - $10\frac{1}{56}$

Or thus:

	cwt. q. lbs.
	212 3 19
	5 $2\frac{1}{7}$
	£212 - 18 - 4·71
10*s*. = £$\frac{1}{2}$	106 - 9 - 2·35
2*s*. = £$\frac{1}{10}$	21 - 5 - 10·07
1*s*. = $\frac{1}{2}$ of 2*s*.	10 - 12 - 11·03
2*d*. = $\frac{1}{6}$ of 1*s*.	1 - 15 - 5·84
Answ.	£353 - 1 - 10·00

Exercises. *Answers.*

	cwt.	q.	lbs.		£	*s*.	*d*.			£	*s*.	*d*.
81.	75	1	16	at	1	9	9	per cwt.	....	112	2	$11\frac{1}{4}$
82.	538	2	17		0	10	4	———	....	278	6	$0\frac{3}{4}$
83.	346	1	4		1	12	7	———	....	564	3	$1\frac{3}{4}$
84.	786	2	8		0	18	9	———	....	737	8	$2\frac{1}{2}$
85.	647	2	11		6	10	8	———	....	4230	19	6
86.	238	0	3		3	19	$7\frac{1}{2}$	———	....	947	12	$10\frac{1}{2}$
87.	181	3	13		2	13	4	———	....	484	19	$6\frac{1}{4}$
88.	251	2	1		1	17	1	———	....	466	6	$9\frac{1}{2}$
89.	103	0	27		5	14	10	———	....	592	15	$6\frac{1}{4}$
90.	418	2	17		2	0	8	———	....	851	5	2
91.	179	3	25		3	11	3	———	....	641	3	1
92.	246	3	24		3	5	4	———	....	806	15	0
93.	319	1	9		0	18	$7\frac{1}{2}$	———	....	297	7	$6\frac{1}{4}$
94.	90	2	10		5	2	$4\frac{1}{2}$	———	....	463	14	1

RULE XI. *In computing the value of acres, roods, and perches*, multiply the perches by $1\frac{1}{2}$ for pence, and the

roods by 5 for shillings, and the result will be the price at £1 per acre.

The *reason of this rule* is evident, since, at £1 per acre, each rood is worth 5*s*., and each perch a fortieth of this, or $1\frac{1}{2}d$.

Exam. 24. What is the yearly rent of 136 acres, 3 roods, 29 perches of land, at £1 - 4 - 3 per acre?

The work, by Rule VIII., in this exercise and some others, is left for the pupil to perform, if it should be thought necessary. The exact answer is £166 - 0 - $6\frac{159}{160}$, as would be found by employing common fractions.

	a.	r.	p.
	136	3	29 at £1 - 4 - 3
		5	$1\frac{1}{2}$
	£136 -	18 -	7·50
4*s*. 0*d*. = £$\frac{1}{5}$	27 -	7 -	8·70
0*s*. 3*d*. = $\frac{1}{16}$ of 4*s*.	1 -	14 -	2·79
Answ.	£166 -	0 -	.6·99, or
	£166 -	0 -	7, nearly.

Exercises. *Answers.*

	a.	r.	p.	£	s.	d.	£	s.	d.
95.	561	2	20 at	1	17	6	1053	0	11
96.	45	2	35	0	16	6	37	14	$4\frac{1}{4}$
97.	77	1	30	0	12	0	46	9	3
98.	586	1	31	1	7	$3\frac{1}{2}$	800	5	$0\frac{1}{4}$
99.	674	2	29	0	11	$4\frac{1}{2}$	383	14	6
100.	311	2	26	1	2	9	354	10	$3\frac{3}{4}$
101.	1268	3	17	1	6	6	1681	4	$8\frac{1}{4}$
102.	139	3	39	2	7	10	334	16	$4\frac{1}{2}$

RULE XII. (1.) *In computing the price of tons, hundreds, and quarters, at £1 per ton,* take the tons and hundreds as pounds and shillings, and multiply the quarters by 3 for pence. (2.) *In computations for troy weight, at £1 per ounce,* take the ounces as pounds, the penny-weights as shillings, and half the grains as pence. (3.) *In computing the prices of yards, quarters, and nails, at £1 per yard,* take each quarter at 5*s*. and each nail at 1*s*. 3*d*.

The *reason of this rule* depends on the obvious principle, that, at £1 per ton, every hundred would cost a shilling, and every quarter 3*d*.: that, at £1 per ounce, a penny-weight would cost a shilling, and a grain a halfpenny: and that, at £1 per yard, a quarter would cost 5*s*. and a nail 1*s*. 3*d*.

Exercises.				£	s.	d.		*Answers.* £	s.	d.
103.	175 tons	18 cwt.	1 qr. at	38	13	0 per ton	...	6799	0	$4\frac{1}{4}$
104.	219 tons	16 cwt.	3 qrs.	11	7	6 per ton	...	2500	13	$0\frac{1}{4}$
105.	93 oz.	7 dwt.	15 grs.	0	10	4 per oz.	...	48	4	$11\frac{1}{4}$
106.	263 oz.	16 dwt.	9 grs.	0	11	3 per oz.	...	148	7	$11\frac{1}{2}$
107.	58 yds.	3 qrs.	1 nail	0	12	8 per yard	...	37	4	$11\frac{1}{2}$
108.	105 yds.	2 qrs.	3 nails	2	8	4 per yard	...	255	8	$2\frac{3}{4}$

The preceding are the most useful applications of the principle on which the ninth and the succeeding rules are founded. Various others might be given, which, however, are omitted, as they are of less importance, and as they will present no difficulty to the pupil who is well acquainted with those already explained.

RULE XIII. In many calculations, instead of multiplying the quantity by the price, it is better to *multiply the price by the quantity.* This is often the case, when compound multiplication can conveniently be employed.

Exam. 25. Required the price of 12 cwt. 3 qrs. 8 lbs. of hops, at £23 - 18 - 6 per cwt.

	cwt. q. lbs.	£ s. d.		
	12 3 8 at	23 18 6 per cwt.		
		12		
		£287 - 2 - 0	= price of 12 cwt.	
2 qrs. = $\frac{1}{2}$ cwt. ...		11 - 19 - 3	= ———	2 qrs.
1 qr. = $\frac{1}{2}$ of 2 qrs.		5 - 19 - $7\frac{1}{2}$	= ———	1 qr.
8 lbs. = $\frac{1}{7}$ of 2 qrs.		1 - 14 - $2\frac{1}{7}$	= ———	8 lbs.
	Answ.	£306 - 15 - $0\frac{9}{14}$	= ———	12 cwt 3 q. 8 lbs.

Exercises.				£	s.	d.		*Answers.* £	s.	d.
109.	8 cwt.	2 qrs.	12 lbs. at	1	15	9	....	15	7	$8\frac{1}{2}$
110.	8 acres	3 roods	19 perches	0	18	10	....	8	7	$0\frac{1}{4}$
111.	9 tons	13 cwt.	....	5	19	8	....	57	14	$9\frac{1}{2}$
112.	11 acres	1 rood	23 perches	1	3	$7\frac{1}{2}$	....	13	9	$2\frac{1}{4}$
113.	1 yard	3 qrs.	2 nails	1	9	3	....	2	14	$10\frac{1}{8}$

114. What is the importation duty on 359 gallons of port wine, at 7*s.* 7*d.* per gallon? *Answ.* £136 - 2 - 5.

115. What is the duty on 710 gallons of Madeira wine, at 7*s.* $8\frac{1}{2}$*d.* per gallon? *Answ.* £273 - 12 - 11.

116. Required the amount of the duty on 124 gallons of French wine, at 11*s.* $4\frac{1}{2}$*d.* per gallon. *Answ.* £70 - 10 - 6.

117. Required the duty on a cwt. of opium, at 8*s.* 9*d.* per lb. *Answ.* £49.

118. Find the duty on 47 cwt. 0 qr. 19 ℔s. of mother-of-pearl shells, at £4 - 13 - 4 per cwt.; and on 197 ℔s. of tortoise-shell, at 3*s*. 11½*d*. per ℔. *Answ*. £220 - 2 - 6, and £38 - 19 - 9½.

119. What is the duty on 517 ℔s. of East India coffee, at £5 - 2 - 8 per cwt.? *Answ*. £23 - 13 - 11.

120. Required the duty on 179 cwt. 2 qrs. 12 ℔s. of Muscatel raisins, at £2 - 3 - 6½ per cwt. *Answ*. £391 - 0 - 4¾.

121. Find the prices of 13 barrels, 17 stones, 9 ℔s. of wheat, at £1 - 7 - 6 per barrel; of 17 barrels, 11 stones, 5 ℔s. of barley, at 14*s*. 6*d*. per barrel; of 15 barrels, 11 stones, 2 ℔s. of oats, at 10*s*. 8*d*. per barrel; and of 11 barrels, 10 stones, 11 ℔s. of malt, at 19*s*. 6*d*. per barrel: the barrel of wheat consisting of 20 stones; of barley, of 16 stones; of oats, of 14 stones; and of malt, of 12 stones. *Answ*. £19 - 1 - 9; £12 - 16 - 9½; £8 - 8 - 5¾; and £11 - 12 - 0¼.

122. What cost 4 dozen and 5 bottles of wine, at 28*s*. per dozen; and 7 bottles, at 78*s*. per dozen? *Answ*. £6 - 3 - 8, and £2 - 5 - 6.

123. What is the rent of 2 roods, 17 perches of meadow, at 7 guineas per acre? *Answ*. £4 - 9 - 1½.

124. Find the price of 2 qrs. 13 ℔s. of sugar, at £2 - 12 - 6 per cwt.; and also of a sugar loaf, weighing 12 ℔s. 10 oz., at 9½*d*. per ℔. *Answ*. £1 - 12 - 4, and 10*s*.

125. What cost 17 cwt. 3 qrs. of hay, at 5 guineas per ton? *Answ*. £4 - 13 - 2¼.

126. Find the prices of 16½ ℔s. of beef, at 6¾*d*. per ℔.; 12¼ ℔s. of mutton, at 6¼*d*. per ℔.; and 9 ℔s. 6 oz. of salmon, at 2*s*. 4*d*. per ℔. *Answ*. 9*s*. 3¼*d*., 6*s*. 4½*d*., and £1 - 1 - 10½.

127.* *Belfast*, 13*th Oct*., 1841.

Mr. Henry Wilson,

Bought of Samuel Jones

725 yards fine white linen, at 4*s*. 1½*d*.	£
87 yards cambric, at 12*s*. 10*d*.........................	
417 yards muslin, at 3*s*. 9½*d*.	
	£284 - 8 - 3

128.

Joseph O'Reilly, Esq.

1841.	To David White & Co.	*Dr*.
Sept. 24*th*,	To fine scale sugar, 4 cwt. 3 qrs. 22 ℔s. at £4 - 17 - 4 per cwt.	£
Dec. 1*st*,	To tea, for 1 chest, containing 83 ℔s. at 7*s*. 4*d*. per ℔	
		£54 - 10 - 1½

129. Dublin, 8th December, 1842. Mr. William Joyce buys from Patrick M'Neale 138 gallons port wine, at 16*s*. 8*d*. per gallon; 130 gallons sherry, at 16*s*. 4*d*.; 110 gallons Madeira, at 26*s*. 9*d*.;

* In this exercise, and the five following, which are called *Bills of Parcels*, the prices of the several articles are to be found and set in the blank spaces towards the right hand: the sum of these partial amounts is the entire amount required. The form of the first of these exercises is that which is usually employed when all the articles are bought at the same time; but when the times are different, the form is generally that of the next exercise. The remaining exercises of this kind are left for the pupil to write out, in proper form, for his own improvement.

and 120 gallons Teneriffe, at 14*s.* 10*d.* Required the whole amount. *Answ.* £457 - 5 - 10.

130. Mr. Edward Stone buys of Hugh Sinclaire of Cork, Jan. 8th, 1842, 156 tierces prime beef, at £5 - 9 - 8; 313 barrels ditto, at £3 - 5 - 8; Feb. 4th, 93 barrels prime pork, at £3 - 8 - 3; Feb. 24th, 64 barrels inferior ditto, at £3 - 3 - 6. Required the entire amount. *Answ.* £2403 - 12 - 11.

131. Belfast, Jan. 10th, 1842. Mr. Alexander Jefferson buys from William Fitzpatrick 218 yards of linen, at 3*s.* 2*d.* per yard; 173 yards of muslin, at 1*s.* 4½*d.* per yard; 2 pieces of printed calico, containing 56½ yards, at 1*s.* 2*d.* per yard; and one piece ditto, containing 27¾ yards, at 10*d.* per yard: and he pays in part £32 - 12 - 6. How much remains due? *Answ.* £18 - 4 - 9.

132. Mr. Robert Bellingham buys from John Cunningham of Glasgow, Feb. 5th, 1842, 3 cwt. of fine scale sugar, at 8*s.* 10*d.* per stone; 2 cwt. coarse ditto, at 7*s.* 4*d.* per stone; a chest of tea containing 86 lbs., at 4*s.* 6*d.* per lb.; 38 gallons of whiskey, at 6*s.* 10*d.* per gallon; and 10 gallons of rum, at 13*s.* 6*d.* per gallon. Required the amount. *Answ.* £55 - 11 - 0.

The method of aliquot parts in its application in finding prices having been fully developed and illustrated in the preceding pages, it may be proper to conclude this article with some miscellaneous matter that could not, with propriety, have been intermixed with the general principles already exemplified.

Operations in the rule of proportion may often be abbreviated by the method of aliquot parts, whether the first term is a unit or not. Thus, if it were proposed to find how much flour might be bought for £6 - 5 - 8, if 7 cwt. 2 qrs. 16 lbs. cost £11; the terms being arranged in the usual way, we may multiply the third term by £6, and take parts of it for 5*s.* 8*d.* By this means, the product of the second and third terms is found to be 48 cwt. 0 qr. $2\frac{8}{15}$ lbs.; and this being divided by 11, by compound division, the quotient is 4 cwt. 1 qr. 13 lbs. nearly, the quantity required.

	£	£	*s.*	*d.*	c.	q.	lbs.
As	11 :	6	5	8 ::	7	2	16
							6
					45	3	12
4*s.* 0*d.* = £$\frac{1}{5}$					1	2	$3\frac{1}{5}$
1*s.* 8*d.* = £$\frac{1}{12}$					0	2	$15\frac{1}{3}$
				11)	48	0	$2\frac{8}{15}$
				Answ.	4	1	13, nearly.

Thus, also, if it be required to find the price of 365 bottles of wine, at £2 - 13 - 6 per dozen, the work will stand as in the margin. Experience will enable the student to judge when this method may be employed with advantage, and when the common method is preferable.

	b.	b.	£	*s.*	*d.*
As	12 :	365 ::	2	13	6
		2			
		730			
10*s.* 0*d.* = £$\frac{1}{2}$		182 -	10 -	0	
2*s.* 6*d.* = $\frac{1}{4}$ of 10*s.*		45 -	12 -	6	
1*s.* 0*d.* = $\frac{1}{10}$ of 10*s.*		18 -	5 -	0	
	12)	976 -	7 -	6	
	Answ.	£81 -	7 -	$3\frac{1}{2}$.	

As an application of this method to quantities of another kind, let it be required to find the sun's mean apparent motion in 10 days, 7 hours, 20 minutes, the mean space which he apparently describes each day in the ecliptic, being 59′ 8″·3. Here, the daily space being multiplied by 10 by compound multiplication, the product, 9° 51′ 23″, is the space described in 10 days. Then, for 6 hours, a fourth of the daily space is taken; for one hour, a sixth of the result; and for 20 minutes, a third of this last result. The sum of all these partial results is 10° 9′ 27″·2 nearly, the mean space required.

		59′	8″·3
			10
h. m.	9°	51	23
6 0 = $\frac{1}{4}$ day		14	47·1
1 0 = $\frac{1}{6}$ of 6 h.		2	27·8
20 = $\frac{1}{3}$ of 1 h.		0	49·3
Answ.	10	9	27·2

The following is another example of the application of this principle. When the radius of a circle is 1, the half of the circumference is 3·14159265; hence, let it be required to find the length of a part of the circumference containing 38° 44′ 33″.

Here, half the circumference being 180°, the reason of the process is obvious; and the answer is true, except the last figure, which should be 8, the difference being occasioned by the rejection of remainders in the divisions.

				3·14159265
30°	0′	0″	= $\frac{1}{6}$ of 180°	·52359877
6	0	0	= $\frac{1}{5}$ of 30°	·10471975
2	0	0	= $\frac{1}{3}$ of 6°	·03490658
	40	0	= $\frac{1}{3}$ of 2°	·01163553
	4	0	= $\frac{1}{10}$ of 40′	·00116355
		30	= $\frac{1}{8}$ of 4′	·00014544
		3	= $\frac{1}{10}$ of 30″	·00001454
			Answ.	·67618416

Various abbreviated modes of finding prices in particular cases are given in works on mercantile arithmetic. Several of these may be useful for the pupil who has had considerable practice in arithmetical calculations, and who is well acquainted with the more common and general rules for such purposes; but they are not fitted for the less experienced pupil, as, from their variety and want of connexion, they tend to perplex and puzzle him, and to make him conceive the subject to be more difficult than it is. No abbreviations, therefore, except such as are of a general and obvious nature, and such as are likely to be most frequently useful, have been introduced in the preceding part of this article. A few of a different kind are here presented, to which the attention of the pupil may be directed, or not, as the teacher may reckon best.

To find the price of any number of articles, at 2d. each: Since $2d. = £\frac{1}{120}$, we divide by 120: then, since $2d. = \frac{1}{6}s.$, we take one sixth of the remainder, regarded as shillings, for the remaining part of the answer. This can all be done in a single line, the division by 120 being performed by conceiving the last figure to be cut off, and then dividing by 12. In like manner the price at 3*d.*, 4*d.*, or 6*d.*, may be found.

	4532 at 2*d.*
$2d. = £\frac{1}{120} = \frac{1}{6}$ of 1*s.*	...£37 - 15 - 4

To find the price at 17*s.* 4*d.:* Find the price at 16*s.*, by the method given in page 206; and for the price at 1*s.* 4*d.* take a twelfth of the result: and *to find the price at* 14*s.* 8*d.*, from the price at 16*s.*, take a twelfth of itself. In like manner, *to find the price at* 15*s.* 2*d.*, to the price at 14*s.*, add a twelfth of itself; and *to find the price at* 12*s.* 10*d.*, from the price at 14*s.*, take a twelfth of itself: and, lastly, *to find the price at* 19*s.* 6*d.*, to the price at 18*s.*, add a twelfth of itself; and *to find the price at* 16*s.* 6*d.*, from the price at 18*s.*, take a twelfth of itself.

3729		at 17*s.* 4*d.*
8		
2983	4	for 16*s.* 0*d.*
248	12	— 1*s.* 4*d.*
£3231	16	

The price at 5*s.* 7½*d.* may be found by adding to the price at 5*s.* an eighth of itself; *the price at* 3*s.* 1½*d.*, by adding to the price at 2*s.* 6*d.* a fourth of itself; and *the price at* 2*s.* 9¾*d.*, by adding to the price at 2*s.* 6*d.* an eighth of itself. We may find *the prices at* 4*s.* 4½*d.*, 1*s.* 10½*d.*, and 2*s.* 2¼*d.*, by the same rules, only *subtracting* instead of adding.

To find the price at £6 - 15, to the price at £6, add an eighth of itself; and *to find the price at* 6*s.* 9*d.*, to the price at 6*s.*, add one eighth of itself. In like manner, *to find the price at* £3 - 7 - 6, to the price at £3, add an eighth of itself; and *to find the price at* £2 - 12 - 6, take an eighth of itself from the price at £3.

The price of 24 *articles* may often be found very readily by taking each penny in the price as 2*s.*; *the price of* 48, by taking each half-penny as 2*s.*; and the price of 96 by taking each farthing as 2*s.* Hence, the price of 25 yards, at 3*s.* 6*d.*, is readily found to be £4 - 7 - 6; for in 3*s.* 6*d.* there are 42 pence; and, by doubling the last figure of this, we have £4 - 4, the price of 24 yards; to which 3*s.* 6*d.* being added, the sum, £4 - 7 - 6, is the required price.

Because 112 farthings = 2*s.* 4*d.*, *to find the price of* 112 *articles*, reduce the price of one to farthings; and doubling the last figure for shillings, take the rest as pounds, and to the result add a sixth of itself. Thus, since in 9¾*d.* there are 39 farthings, to find the price of 112, at 9¾*d.* each, by doubling the last figure of 39, we have £3 - 18, to which a sixth of itself being added, we have £4 - 11, the price required.

To find the price of 120, reckon every penny in the given rate 10*s.* Thus, 120, at 4*d.* each, amount to 40*s.*, or £2; and at 5½*d.* to 55*s.*, or £2 - 15. The same results would be obtained by taking the farthings in the given rate as pounds, and dividing by 8.

Almost numberless other abbreviations might be added. In addition to those given above, others will be found in the article on mental arithmetic.

TARE AND TRET.

The whole weight of any commodity, together with the weight of the box, barrel, &c., which contains it, is called its GROSS WEIGHT.

The weight of the box, barrel, &c., which contains any commodity, is called its TARE.

The SUTTLE WEIGHT is what remains after the tare is deducted.

TRET is an allowance of 4 lbs. for each 104 lbs. of the suttle, in goods that are liable to waste.

CLOFF is an old-fashioned name for an allowance of 2 lbs. which used to be made in every 3 cwt. of what remains after the other deductions are made. This allowance was for the turning of the beam in weighing the goods. Nowadays balances are so good that no such allowance is needed.

The REAL OR NET WEIGHT is what remains after all deductions are made.

The arithmetical working of allowances and deductions such as those described in Tare and Tret is so simple that neither rules nor examples are needed.

COMPOUND PROPORTION.

RULE I. (1.) By the rule for simple proportion, find a fourth proportional to two given terms of the same kind with one another, and to the term which is of the same kind as the answer. (2.) To two other terms of the same kind, and to the proportional last obtained find a fourth proportional; and thus proceed if there be more terms: the final result will be the answer.

RULE II. (1.) Place the term which is of the same kind as the required term, in the last place. (2.) Comparing the other given terms by pairs, place each as antecedent or consequent, according to the general rule for simple proportion. (3.) Divide the continual product of all the consequents and the last term, by the

continual product of all the antecedents; the quotient will be the answer.

As a contraction in the use of this latter rule, divide an antecedent, and either the last term, or any consequent, by any number that will divide them without remainders, and employ the results instead of those terms: or, if an antecedent and any consequent, or an antecedent and the last term be the same, reject them.

Exam. 1. If 40 gallons of ale serve 17 persons 5 days, how many gallons will 9 persons use in a year, at the same rate?

I. As 5 days : 365 days :: 40 gallons : 2920 gallons.
As 17 persons : 9 persons :: 2920 gallons : $1545\frac{15}{17}$ gallons.

In this operation, we first proceed as if the number of persons in both cases were 17, and on this supposition we find that 2920 gallons would serve those persons for a year. But the number of persons being 9, instead of 17, we find by the second analogy, that if 17 persons would use 2920 gallons in a year, 9 persons would use only $1545\frac{15}{17}$ gallons in the same time.

II. $\left.\begin{array}{l}\text{As 5 days : 365 days}\\ \text{17 persons : 9 persons}\end{array}\right\}$ :: 40 gallons : $1545\frac{15}{17}$ gallons.

In this method, that term is placed last which is of the same kind as the required term. Then were the number of persons the same, the answer would evidently be greater than 40 gallons; and therefore we put 365 days in the second, and 5 days in the first place: but were the number of days the same, the required term would be less than 40 gallons; wherefore, we put 9 persons in the second place, and 17 in the first. We then multiply the product of the consequents by the common third term, 40, and divide the result by the product of the antecedents, and the same answer is found as before. This operation is, in effect, the same as that in the first method; for the result of the first single analogy, without the actual multiplication and division, is $\frac{365 \times 40}{5}$, and, consequently, the second analogy becomes $17 : 9 :: \frac{365 \times 40}{5} : \frac{9 \times 365 \times 40}{5 \times 17}$; whence it appears, that, in both methods, the same multiplications and divisions are in reality performed, and consequently the one is only a modification of the other. In this method 5 and 365, or 5 and 40, might be divided by 5 as a contraction.

One very considerable advantage belonging to the second rule is, that by it the operation is kept free from fractions till the conclusion; while, in the other mode, fractions often arise from the first analogy, and render the remaining part of the work more intricate and difficult.

Exam. 2. If 15 men, working 12 hours daily, reap 60 acres in 16 days, in what time would 20 women, working 10 hours daily,

reap 98 acres, 7 men being able to reap as much as 8 women, in the same time?

$$\left.\begin{array}{lll} \text{As } 20 \text{ persons} & : & 15 \text{ persons} \\ 10 \text{ hours} & : & 12 \text{ hours} \\ 60 \text{ acres} & : & 98 \text{ acres} \\ 7 & : & 8 \end{array}\right\} :: 16 \text{ days} : 26\tfrac{22}{25} \text{ days.}$$

In this example, since the answer is to be in time, we place 16 days as the common third term. Then, it is evident, that, were all things alike, except the number of workers, the time required would be less than 16 days, 20 persons requiring less time than 15; and therefore 15 is put as consequent, and 20 as antecedent. As, however, the women work only 10 hours daily, while the men work 12, the required time would be longer on this account; therefore 12 is put in the second column, and 10 in the first. The other terms are arranged on similar principles; and the answer is found by dividing the continual product of 16 and the consequents, by the continual product of the antecedents.

The work may be shortened, as in the margin, by dividing 20 and 15 by 5, and 10 and 12 by 2; then, by omitting 7, and writing 14 instead of 98, as 98 is equal to 14 times 7. For similar reasons, we omit 6, and write 10 instead of 60: we omit also 4, and write 2 in place of 8; and, lastly, this 2 is omitted, and the antecedent 10 is changed into 5. We then divide 672, the continual product of 3, 14, and 16, by 25, the product of 5 and 5, and we obtain the same answer, as before.

$$\left.\begin{array}{rcl} (4) & : & 3 \\ 5 & : & (6) \\ 5(10)(60) & : & 14 \\ & & (8)(2) \end{array}\right\} :: 16 : 26\tfrac{22}{25}.$$

This question might also have been wrought by means of four operations in simple proportion. All these, however, except one, would give origin to fractions, the neglecting of which would prevent the true answer from being found. In every respect, therefore, the second rule is greatly preferable.

Exer. 1. If the carriage of 59 cwt., 19 miles, cost £2 - 16, how far may 43 cwt. be carried, at the same rate, for £2 - 4? *Answ.* $20\frac{291}{602}$ miles.

2. If the carriage of 13 cwt., 65 miles, cost £2 - 5, how many hundreds may be carried 40 miles, at the same rate, for £3 - 15? *Answ.* $35\frac{5}{24}$ cwt.

3. If 12 horses plough 11 acres in 5 days, how many horses would plough 33 acres in 18 days? *Answ.* 10.

4. If a man walking 12 hours each day, travel 250 miles in 9 days, in how many days, walking 10 hours each, at the same rate, would he travel 400 miles? *Answ.* $17\frac{7}{25}$ days.

5. If the expenses of a family of 8 persons, amount to £42 in 16 weeks, how long will £100 support a family of 6 persons, at the same rate? *Answ.* $50\frac{50}{63}$ weeks.

6. If 29 men, in 5 days of 12 hours each, reap 32 acres, in how many days of 13 hours each, will 20 men, working equally, reap 40 acres? *Answ.* $8\frac{19}{52}$ days.

7. If £15 - 12 pay 16 labourers for 18 days, how many labourers, at the same rate, will £35 - 2 pay for 24 days? *Answ.* 27.

8. If 36 yards of cloth, 7 quarters wide, cost £25 - 4, what cost 120 yards of the same quality, but only 5 quarters wide? *Answ.* £60.

9. If a tradesman earn 16 guineas in 108 days, how many sovereigns would he earn, at the same rate, in 270 days; 20 guineas being equivalent to 21 sovereigns? *Answ.* 42.

10. If the rent of a farm of 26 a. 2 r. 23 p. be £50 - 8 - 9, what would be the rent of another, containing 17 a. 3 r. 2 p., if 6 acres of the latter be worth 7 of the former? *Answ.* £39 - 4 - 7.

11. If a puncheon of rum containing 85 gallons, cost £58 - 8 - 9, what would be the value of a hogshead containing 63 gallons, and composed of four parts of the same rum, and one part of water? *Answ.* £34 - 13 - 0.

12. In what time would 23 men reap a field which 40 women would reap in 6 days, if 7 men can reap as much as 9 women? *Answ.* $8\frac{8}{69}$ days.

13. If a person, walking 13 hours each day, travel 191 miles in 7 days; in how many days of 10 hours will he complete the remainder of a journey of 500 miles, at the same rate each hour? *Answ.* $14\frac{1379}{1910}$.

14. If 63 lbs. of tea cost £20 - 10 - 6, what cost 70 lbs. of a different quality, 9 lbs. of the former being equal in value to 10 lbs. of the latter? *Answ.* £20 - 10 - 6.*

INTEREST.

The sum to be paid by a person for the use of money which he owes, is called the INTEREST of that money.

The money due is called the PRINCIPAL.

The sum of the principal and interest is called the AMOUNT.

The RATE is the money allowed for the use of one hundred pounds for any given time, but usually for a year.

When interest is charged on the *original principal only*, it is termed SIMPLE INTEREST.

When interest is charged, not only on the original principal, but also *on the interest as it becomes due*, it is called COMPOUND INTEREST.

* Many of the questions usually proposed under the head of compound proportion are quite misplaced, as some of them are merely anticipations of what is afterwards delivered in *interest*, and others belong to *mensuration*, or other subjects, with the principles of which the pupil is supposed to be unacquainted. Hence, the number and variety of exercises here given, are purposely less than in several other works on arithmetic.

It is scarcely necessary to remark that *per cent.*, a contraction for *per centum*, means *by the hundred*; and that *per annum* means *by the year.*

SIMPLE INTEREST.*

RULE I. *To find the interest of a given sum for a year, at a given rate per cent. per annum:* Multiply the principal by the rate, and divide the product by 100.† Or,

As £100 is to the rate per cent. per annum, so is the principal to its interest for one year.

Exam. 1. Required the interest of £576 - 5 - $8\frac{1}{2}$, for 1 year, at 6 per cent. per annum.‡

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* In interest five quantities are concerned, the *principal*, the *rate*, the *time*, the *interest*, and the *amount*; and any three of these, except the principal, the interest, and the amount, being given, the rest can be found. Hence, computations in interest admit of several problems. The most useful, however, is that in which the principal, the time, and the rate are given, to find the interest or the amount. This problem may be resolved, in all cases, by means of the first or second rule: but the third and fourth present modified, and, in many cases, shorter methods of effecting the same.

† The following rule will be found useful: *To divide money* by 100, for the pounds of the quotient, take the pounds of the dividend, except the last two figures, which are to be divided by 5 for shillings: from the remainder, with half the shillings of the dividend annexed, reject a twenty-fifth part, and regard what remains as farthings. If there be pence, or an odd shilling, their effect in modifying the quotient may be estimated as nearly as possible. Thus, let it be required to divide £8947 - 13 - 8 by 100. Here,

£	s.	d.
100)8947	13	8
£89 -	9 -	$6\frac{1}{2}$

by cutting off two figures, we have £89; and one fifth of 47 is 9, the shillings required, and the remainder is 2. This remainder, with 7, the half of 14 shillings, annexed, becomes 27, from which 1, nearly its twenty-fifth part, being rejected, we have 26 farthings, or $6\frac{1}{2}d.$ We use 14 shillings in this example, because 13*s.* 8*d.* is nearly 14*s.*

The *reason* of this process will be understood from the rule given in page 190.

As another example, let it be required to divide £2658 - 16 - 10 by 100. Here, after cutting off two figures, we have £26, and the fifth of the remainder is 11 for shillings, with 3 remaining. This remainder being prefixed to $8\frac{1}{2}$, the half of 17*s.*, to which 16*s.* 10*d.* is nearly equal, we have $38\frac{1}{2}$, the twenty-fifth part of which is obviously about $1\frac{1}{2}$. Then $38\frac{1}{2}$ being diminished by this quantity, the remainder is 37 farthings, or $9\frac{1}{4}d.$, which completes the quotient.

$$\frac{£2658 - 16 - 10}{£26 - 11 - 9\frac{1}{4}}$$

‡ The rate of interest has varied much at different periods, and in different countries, but it has been generally observed to diminish as commerce extends. In Italy, about the beginning of the thirteenth century, it varied between 20 and 30 per cent. per annum; and, in the Netherlands, it was fixed by Charles V. in 1560 at 12 per cent. By an Act of the 37th year of Henry VIII., interest in England was not to exceed 10 per cent. By the 21st of James I. it was reduced to 8 per cent. Soon after the Restoration it was reduced farther, to 6 per cent.; and in the 12th of Anne, to 5 per cent. The legal rate of interest in Ireland was 6 per cent. The Usury Laws imposing these restrictions were entirely repealed in 1854 by 17 & 18 Vic., c. 90.

Or thus, according to second note in page 225.

£576 - 5 - $8\frac{1}{2}$
6
£34|57 - 14 - 3
20
11|54
12
6|51
4
2|04

£576 - 5 - $8\frac{1}{2}$
6
100)3457 - 14 - 3
Answ. £34 - 11 - $6\frac{1}{2}$

Here, by the first, or exact method, the answer is £34 - 11 - $6\frac{1}{2}$, with $\frac{4}{100}$ or $\frac{1}{25}$ of a farthing.

The *reason of the operation* is quite evident, as it is nothing more than this: as the principal, £100, is to its interest, £6, so is the principal, £576 - 5 - $8\frac{1}{2}$, to its interest; and it is evident that, as often as the one principal contains its interest, so often will the other contain its interest: that is, by the nature of proportion, the interest will be proportional to the principal.

Exam. 2. Required the interest and the amount of £619 - 9 - 6, for 1 year, at $5\frac{1}{2}$ per cent. per annum.

£619 - 9 - 6, at $5\frac{1}{2}$ per cent.
$5\frac{1}{2}$
3097 - 7 - 6, for 5 per cent.
309 - 14 - 9, for $\frac{1}{2}$ ——
100)3407 - 2 - 3, for $5\frac{1}{2}$ ——
Add $\begin{cases} 34 - 1 - 5 = \text{interest} \\ 619 - 9 - 6 = \text{principal} \end{cases}$
£653 - 10 - 11 = amount.

The division at full length is as follows:

£34|07 - 2 - 3
20
1|42
12
5|07

In this operation the principal is multiplied by 5, for 5 per cent.; and for $\frac{1}{2}$ per cent. half the principal is added to the product. At the conclusion of the contracted division, it gives the result more nearly true to reject *one*, than nothing, from 21, though less than 25, and more especially as there is 3*d.* in the dividend. In the examples that follow, the division at full length will be omitted; it may be proper, however, for the pupil occasionally to work exercises both ways.

Exam. 3. What is the interest of £1374 - 1 - 9, for 1 year, at $5\frac{5}{8}$ per cent. per annum?

Since $5 = \frac{1}{20}$ of 100, this example might be wrought by taking a twentieth of the principal and increasing the result by an eighth of itself.

£1374 - 1 - 9, at $5\frac{5}{8}$ per cent.
$5\frac{5}{8}$
6870 - 8 - 9
$\frac{5}{8} = \frac{1}{8}$ of 5 858 - 16 - 1
100)7729 - 4 - 10
£77 - 5 - 10, *answ.*

Exercises. Find the interests of the following sums, for 1 year, at the given rates per cent. per annum:—

	Exercises. £	s.	d.		*Answers.* £	s.	d.
1.	774	11	3, at	5	38	14	$6\frac{3}{4}$
2.	15	0	0	$4\frac{1}{2}$	0	13	6
3.	39	12	6	$5\frac{1}{2}$	2	3	7
4.	468	16	8	$3\frac{1}{2}$	16	8	$2\frac{1}{4}$
5.	57	10	0	$4\frac{1}{2}$	2	11	9
6.	876	12	6	$5\frac{3}{4}$	50	8	$1\frac{1}{2}$
7.	9	13	4	$3\frac{3}{4}$	0	7	3
8.	376	12	8	4	15	1	$3\frac{3}{4}$
9.	86	5	0	$7\frac{7}{8}$	6	15	10
10.	637	11	0	$5\frac{5}{8}$	37	3	$9\frac{3}{4}$
11.	17	7	0	$3\frac{1}{2}$	0	12	$1\frac{1}{4}$
12.	899	10	0	$4\frac{3}{4}$	42	14	$6\frac{1}{4}$
13.	37	6	0	$4\frac{1}{2}$	1	13	$6\frac{3}{4}$
14.	534	4	0	$7\frac{1}{4}$	38	14	7
15.	671	19	6	$4\frac{1}{4}$	28	11	2
16.	10 guineas			6	0	12	$7\frac{1}{4}$

Rule II. *To find the interest of a given principal for any other time than a year:* (1.) Find the interest for a year, by Rule I. (2.) As one year is to the given time, so is the interest for one year to the interest required.

The work may frequently be abbreviated by finding the interest for months or other fractional parts of a year, by the method of aliquot parts. In using this method, the answer will often be found with more ease, or with a greater degree of correctness, by multiplying by the rate; then multiplying, or taking aliquot parts for the time; and, last of all, dividing by 100.

Exam. 4. Required the interest of £99 - 2 - 4, for $2\frac{3}{4}$ years, at 4 per cent. per annum.

In this example, the interest for 1 year is first found, which is £3 - 19 - $3\frac{1}{2}$, nearly. This is then multiplied by $2\frac{3}{4}$, the number of years. It might have been done by multiplying by 3, and subtracting a fourth of £3 - 19 - $3\frac{1}{2}$. The formal analogy would have been, as 1 y. : $2\frac{3}{4}$ y. : : £3 - 19 - $3\frac{1}{2}$: £10 - 18 - $0\frac{3}{4}$.

	£99 -	2 -	4	
			4	
100)	396 -	9 -	4	
	£3 -	19 -	$3\frac{1}{2}$,	interest for 1 year.
			$2\frac{3}{4}$	
	7 -	18 -	7	
	1 -	19 -	$7\frac{3}{4}$	
	0 -	19 -	10	
	£10 -	18 -	$0\frac{3}{4}$	

Exam. 5. Required the interest of £179 - 12 - 11, for 1 year and 7 months, at 5 per cent. per annum.

The interest for 1 year is found, by the method already explained, to be £8 - 19 - 7¾. The rest of the work, by aliquot parts, is as follows:—

	£8 - 19 - 7¾ = interest for 1 year
6 months = ½ year	4 - 9 - 9¾ = ——— 6 months
1 month = ⅙ of 6 months	0 - 14 - 11⅝ = ——— 1 month
	£14 - 4 - 5¼ = interest required.

In the second method, the division by 100 is delayed till the end of the operation: everything else is as before. By this method, the error that often arises from neglecting the remainders in the division by 100, is done away, and the answer thus found is more nearly true, than that which, in many cases, would be obtained by the other method.*

Or thus:

	£179 - 12 - 11
	5
	898 - 4 - 7
6 months = ½ year	449 - 2 - 3½
1 month = ⅙ of 6 months	74 - 17 - 0½
100)	1422 - 3 - 11
	£14 - 4 - 5¼

Exercises. Find the interests of the following principals, for the given times, and at the given rates per cent. per annum:—

	Exercises. £	s.	d.		*Answers.* £	s.	d.
17.	81	10	0	for 2 y. 5 m. at 4¾	9	7	1¼
18.	24	0	0	1 y. 4 m. at 5¼	1	13	7¼
19.	419	7	9	1 y. 10 m. at 4⅜	33	12	9¼
20.	493	16	8	1 y. 8 m. at 6	49	7	8
21.	24	18	9	10 m. at 6	1	4	11¼
22.	427	8	8	1 y. 5 m. at 5¾	34	16	4¼
23.	92	12	0	1 y. 10 m. at 6½	11	0	8¼
24.	25	0	0	1 y. 9 m. at 5	2	3	9
25.	651	0	0	7 m. at 4½	17	1	9¼
26.	584	18	8	1 y. 9 m. at 3⅛	31	19	9¼
27.	4	7	6	5 m. at 6	0	2	2¼
28.	50 guineas			1 y. 2 m. at 6	3	13	6

Exam. 6. Required the interest of £342 - 11 - 8, for 86 days, at 4 per cent. per annum.

In this exercise, the interest for a year is found (by Rule I.) to be £13 - 14 - 0¾, nearly; and as 365 days : 86 days :: £13 - 14 - 0¾ : £3 - 4 - 7, nearly, the interest required.

* *The interest of a sum for any number of months, at 6 per cent. per annum,* may be very easily found by multiplying the sum by half the number of months, and dividing the result by 100: and hence the interest at other rates may be derived by means of aliquot parts. Thus, to find the interest of £250 for 10 months, at 4 per cent. per annum, we multiply £250 by 5; and dividing the product by 100, we find the interest at 6 per cent. to be £12 - 10 - 0. We then take from this a third of itself, and the remainder is £8 - 6 - 8, the interest required. The *reason* of this is plain, since the number of pounds in the rate (£6) is half the number of months in the year. Other contractions in the computation of interest will be found in the article on mental arithmetic.

Exam. 7. Required the interest of £429 - 17 - 4, from June 29, 1841, till Feb. 12, 1842, at 5 per cent. per annum.

The number of days from the 29th of June, till the 12th of February following, is found, by the method shown in page 72, to be 228; and the interest of £429 - 17 - 4 for a year, computed (by Rule I.) to be £21 - 9 - 10½. Then, as 365 days : 228 days :: £21 - 9 - 10½ : £13 - 8 - 6¼, the interest required.

It will readily be seen, that this and all similar questions may be wrought by compound proportion. The terms will be arranged thus:—

$$\left.\begin{array}{l}\text{As £100 : £5}\\ \text{365 days : 228 days}\end{array}\right\} :: \text{£429 - 17 - 4 : £13 - 8 - }6\tfrac{1}{4}, \textit{answ.}$$

Exercises. Find the interests of the following sums, for the proposed times, and at the assigned rates per cent. per annum :—

	Exercises.				*Answers.*		
	£	s.	d.	days.	£	s.	d.
29.	456	10	0	for 31, at 5	1	18	9¼
30.	1000	0	0	— 1, — 6	0	3	$3\frac{21}{73}$
31.	700	0	0	— 89, — 5	8	10	8¼
32.	381	1	8	—264, — 6	16	10	9
33.	447	12	6	from July 8, till Dec. 26, at 5¾	12	1	2
34.	61	0	0	—— Ap. 21, — Sept. 4, — 5½	1	5	0
35.	43	3	0	—— June 20, — Nov. 8, — 6	1	0	0
36.	943	1	8	—— May 1, — Oct. 21, — 5¼	23	9	4
37.	40	10	0	—— Aug. 24, — Jan. 1, — 6	0	17	3¾

38. Required the interest of 15 guineas from March 17, 1840, till January 26, 1842, at 6 per cent. per annum 1 15 2½

39. What is the interest of £53 - 6 - 8, from June 14, 1841, till Sept 22, 1843, at 4½ per cent. per annum 5 9 1¾

RULE III. *To find the interest of a given sum for any number of days:* Multiply the principal by twice the rate, and the product by the days, and divide the result by 73,000.

The division by 73,000 may be performed by the following rule :— Below the dividend write one third of itself, one tenth of that third, and one tenth of that tenth, rejecting shillings and remainders: then add the four lines together, divide the sum by 100,000 (or cut off five figures), and reject a farthing for each £10 in the result.*

* The exact correction is a farthing for £10 - 8 - 4¼. The *reason* of this process will appear from performing the operation indicated in it on 73,000, the result being 100,010. Now 10, the excess of this above 100,000, is contained in it 10,001 times, and 10,001 farthings are £10 - 8 - 4¼.

The tenths will be obtained by setting the figures one place to the right hand, and rejecting the last of them.

Exam. 8. What is the interest of £372 - 10 - 10, from February 12, till December 17, 1840, at $4\frac{1}{2}$ per cent. per annum?

1840 being a leap year, the number of days is found to be 309; and the product of this, of the given principal, and of 9 (twice $4\frac{1}{2}$), being found, as in the margin, is £1036077, which being divided by 73,000, the quotient is £14 - 3 - $10\frac{1}{4}$, the interest required. The division by 73,000, by the second mode, will stand as in the margin, below the answer as found by the first method. After cutting off five figures, we have £14 remaining to the left. Then, according to the method shown in page 225, we divide the number expressed by the next two figures by 5, and obtain for quotient 3 shillings, with the remainder 4. Prefixing this to the next figure, we have 44; and rejecting 2, for a twenty-fifth of 44, we have remaining 42 farthings, or $10\frac{1}{2}d$. We then correct the result by rejecting one farthing, and we find the same result as before.

```
          £372 - 10 - 10
                       9
         £3352 - 17 -  6, or
         £3353 -  0 -  0 nearly
           309
         30177
       10059
73000)1036077
      £14 - 3 - 10¼, answ.

          1036077
 1/3 ......  345359
 1/10 .....   34536
 1/10 .....    3453
          14,19425
        £14 - 3 - 10½
 Correc. ......   0¼
        £14 - 3 - 10¼, answ.
```

The reason of the rule will be evident from the operation by compound proportion, if, instead of £100 and the rate per cent., their doubles be employed. Thus, we should have, in this exercise:—

$$\left.\begin{array}{l}\text{As £200 : £9}\\ \text{365 days : 309 days}\end{array}\right\} :: \text{£}372 - 10 - 10 : \text{£}14 - 3 - 10\tfrac{1}{4};$$

and, in working this by the rule for compound proportion, we should multiply together the principal, the days, and twice the rate, and divide the product by 365 × 200, or 73000.

When the rate is 5 per cent., since the double of 5 is 10, we merely divide the product of the principal and the days by 7300.

	Exercises. £	s.	d.			*Answers.* £	s.	d.
40.	648	15	6	from June 2, till Nov. 25, at 5	......	15	12	10
41.	14	0	0	—— Mar. 23, — Nov. 2, — 6	......	0	10	$3\frac{3}{4}$
42.	688	18	4	—— Mar. 10, — Aug. 25, — 6	......	19	0	6
43.	884	8	8	—— Mar. 3, — Oct. 28, — 5	......	28	19	$1\frac{1}{2}$
44.	4868	15	0	—— June 8, — Nov. 1, — $6\frac{1}{2}$	......	126	11	9
45.	66	8	0	—— May 6, — Aug. 21, — $5\frac{1}{2}$	......	1	1	5

46. Required the interest of £14 for 3 years and 122 days, at 6 per cent. per annum 2 16 0

The following rule will be found easy and useful in interest and discount:—

RULE IV.* *To find the interest of a given principal for any number of days, at 4 per cent. per annum:* (1.) Multiply the principal by the days: (2.) to the product add one tenth of itself: (3.) from the sum take four times the same product, wanting the last three figures: (4.) divide what remains by 10,000 (or cut off four figures); the quotient will be the answer nearly. (5.) When the interest is large, reject a farthing for each £10 contained in it.

For other rates than 4 per cent. increase or diminish the product of the principal and days, by the method of aliquot parts, and then proceed by the rule.

Exam. 9. Required the interest of £8985 - 14, for 12 days, at 4 per cent. per annum.

Here the product of the principal and days is £107828 nearly, and the tenth of this (found by setting each figure one place nearer the right-hand side, and increasing the unit figure by 1, because 28 is nearly 30) being added to it, the sum is 118611. After this, we multiply 107 by 4, and increase the product by 3 (carried for 4 times 8, the first of the figures cut off). The result, 431, is then subtracted, and the remainder, 118,180, divided by 10.000, in the way pointed out in the last example. The quotient is £11 - 16 - 4¼; from which, because it is nearly £10, a farthing is subtracted, and the remainder, £11 - 16 - 4, is the interest required. Had the rate been 5 per cent. we must have increased £107828 - 8, by one fourth of itself, and then have added to the result one tenth of itself, &c.

```
£8985 - 14
        12
----------
107828 -  8
 10783
----------
118611
   431
----------
11,8180
----------
£11 - 16 - 4¼
             ¼
----------
£11 - 16 - 4, answ.
```

With respect to the *reason* of this easy and expeditious rule, the reader who has studied decimal fractions will find, by the last rule, that the interest of £1, for 1 day, at 4 per cent. per annum, is £8 ÷ 73000, or £·0001096, nearly, or £·00011 − £·0000004, nearly: and it will appear, on a little consideration, that the operation by the rule is nothing else than multiplying by ·00011 and ·0000004,

* The following method of *finding interest for days at 6 per cent. per annum* may be found useful, when the interest to be found is not very large:—Divide the product of the principal and days by 100; take one third of the quotient for shillings, and one sixth of the remainder for farthings; and from the sum thus obtained reject a penny for each six shillings contained in it; the remainder will be the interest required, nearly. Another correction may sometimes be made, by adding to the result obtained by the preceding part of the rule, a penny for each six shillings contained in the first correction; or a penny for each £20 in the entire interest will give nearly the same correction.

and taking the difference of the results. The correction is necessary, because the decimal £·0001096 is not the exact interest of one pound for a day.

	Exercises.				*Answers.*		
	£	s.	d.		£	s.	d.
47.	70	6	0	from June 9, till Dec. 6, at 4	1	7	$8\frac{3}{4}$
48.	593	12	6	—— May 12, — Oct. 29, — 4	11	1	$2\frac{1}{4}$
49.	374	5	0	—— April 1, — Dec. 29, — 4	11	3	$1\frac{1}{2}$
50.	247	0	0	—— Mar. 14, — June 8, — 6	3	9	10
51.	30	2	0	—— May 13, — Sept. 29, — 6	0	13	9
52.	176	11	4	—— Mar. 17, — Aug. 25, — $5\frac{1}{2}$	4	5	8

The computation of interest, on *accounts current*, affords a useful application of the preceding principles. An ACCOUNT CURRENT contains a statement of the mercantile transactions of one person with another, when immediate payments are not made. It is usually written on two pages, marked Dr. and Cr. (Debtor and Creditor), in the manner of a *Ledger account*, the left-hand page containing the payments made by the merchant who furnishes the account, and the other what is paid to him. At the foot of this page and the next, there is a specimen of this kind of account; and the following is the method of computing the interest on it, at 4 per cent. per annum:—

			£	s.	d.	days	*Dr.*	*Cr.*
Feb.	10.	To	186	7	6	× 303 ...	56472	
	25.	To	214	10	0	× 288 ...	61776	
June	20.	To	415	8	4	× 173 ...	71867	
March	24.	By	166	13	4	× 261		43500
April	6.	By	347	18	0	× 248		86279
Sept.	26.	By	200	0	0	× 75		15000
							190115	144779
							144779	
							45336	
						$\frac{1}{10}$	4534	
							49870	
						45 × 4 + 1	181	
							4,9689	
							£4 - 19 - $4\frac{1}{2}$, interest.	

Exer. 53. *Dr.* Mr. JOHN JARDINE, Newry, in Account.

1842		£	s.	d.
Feb. 10	To balance by account furnished	186	7	6
25	To amount of sugar........................	214	10	0
June 20	To amount of rum and sugar	415	8	4
Dec. 10	*To interest due on this account*	4	19	$4\frac{1}{2}$
		821	5	$2\frac{1}{2}$

For rightly understanding this computation, it is necessary to consider, that the account is made up till the 10th of December, and the interest calculated on it till that date. We place in a column, as above, all the sums on the debit side, and then all those on the credit side, prefixing to both their dates. To the former, also, we prefix the word *to*, and to the latter *by*, for the sake of distinction. We next find, successively, the number of days between Feb. 10 and Dec. 10, between Feb. 25 and Dec. 10, &c., and place them in the next column. A debit column and a credit one (marked Dr. and Cr.), are then formed, and all the sums on the debit side are multiplied by the corresponding number of days, and the products are placed in the debit column. In like manner, the products of the sums on the credit side, by the days which follow them, are placed in the credit column. The sums of the two columns are then taken, and the debit side is found, by subtraction, to exceed the other side by 45336; which, by means of Rule IV. (or of Rule III.), gives £4 - 19 - 4½, the interest due on the entire account. This is placed on the debit side of the account; and then the sum of all on the credit side is taken from the sum of all on the debit side, and the remainder £106 - 13 - 10½, is placed on the credit side, as the sum due by the person to whom the account is furnished. It is scarcely necessary to say, that the last two lines, in Italics, form the *answer* of the account.

The pupil ought not only to perform the computation of the interest on the following accounts current, but also to write the accounts out, in proper form, on a sheet of paper, after the manner of the specimen given at the foot of this page and the preceding. The answers are in the lines which are printed in Italics.

Exer. 54, 55, 56. Required the principal and interest due on each of the following accounts current, till the date at the end of each, the first at 6, the second at 5, and the third at 6 per cent. per annum.

Dr. Mr. J. Fox, in Account Current with S. Bell. *Cr.*

		£	s.	d.			£	s.	d.
1842					1842				
May 19	To goods	512	12	6	June 13	By cash	400	18	0
Aug. 23	To tea	273	8	0	Nov. 8	By wheat	680	0	0
Oct. 4	To goods	186	10	0	Dec. 1	By bill	73	5	8
Nov. 18	To sugar	272	5	0	1843				
1843 Jan. 18	*To balance of interest* ...	10	4	11	Jan. 18	*By balance to new account*	100	16	9
		£1255	0	5			£1255	0	5

Current with Charles Caulfield, Belfast. *Cr.*

		£	s.	d.
1842				
Mar. 24	By amount of flour	166	13	4
April 6	By cash	347	18	0
Sept. 26	By bill on Cavan & Co., Dublin	200	0	0
Dec. 10	*By balance to your debit in a new account*	106	13	10½
		£821	5	2½

Dr. Mr. C. Johns, London, in Account Current with Thomas Linn & Co., Dublin. *Cr.*

		£	s.	d.			£	s.	d.
1841					1841				
Sept. 3	To balance...	1280	3	11	Sept. 19	By sugar ...	1510	10	0
Dec. 21	To linen......	793	18	0	1842				
1842					Mar. 20	By goods ...	1248	8	0
Jan. 27	To goods ...	1040	5	0	May 25	By bill	912	16	8
Mar. 26	To linen ...	838	14	2	June 26	*By balance to new acct.*	306	16	$3\frac{1}{2}$
June 26	*To balance of interest*	25	9	$10\frac{1}{2}$					
		£3978	10	$11\frac{1}{2}$			£3978	10	$11\frac{1}{2}$

Dr. Mr. T. Hart, in Account Current with H. Orr. *Cr.*

		£	s.	d.			£	s.	d.
1840					1840				
Feb. 9	To linen	768	8	9	April 1	By sales of flaxseed	418	4	8
May 7	To lawn	436	17	6					
July 8	To goods	948	5	10	June 27	By sugar	500	0	0
Aug. 18	To linen	673	11	0	Sept. 11	By bill on Ash & Co., London	1533	12	6
Oct. 29	*To balance of interest*	34	10	10					
					Oct. 29	*By balance to new acct.*	408	16	9
		£2861	13	11			£2861	13	11

The following rules serve for the resolution of the remaining cases of interest. As these cases are of minor importance, the rules are given without proof, and without illustration by examples. They are easily proved, however, by the principles of proportion; and the pupil will find it easy to apply them in the resolution of the subjoined exercises.

Rule V. *To find what principal, in a given time, would produce a given interest, at a given rate per cent. per annum:*

$$\left.\begin{array}{ll}\text{As the rate} & : \pounds 100 \\ \text{The given time} & : 1 \text{ year}\end{array}\right\} :: \text{the interest} : \text{the principal.}$$

Exer. 57. How much money must be lent on the 2nd of April, at 6 per cent. per annum, to bring in for interest £24, on the 18th of November following? *Answ.* £634 - 15 - $7\frac{3}{4}$.

58. What principal, at 5 per cent. per annum, will bring a yearly income of £341 - 5? *Answ.* £6825.

Rule VI. *To find what principal, in a given time, would increase to a given amount, at a given rate per cent. per annum:* (1.) To the product of the time and rate, add the product of £100 and 1 year, in the same name as the given time: (2.) Then, as the sum is to the above-mentioned product of £100 and 1 year, so is the amount to the principal.

Exer. 59. What principal lent on the 1st of January, 1840, at $5\frac{1}{2}$ per cent. per annum, would amount to £1000, on the 29th of Sept. in the same year? *Answ.* £960 - 12 - $6\frac{1}{2}$.

60. What sum must be lent, at simple interest, at 4 per cent. per annum, that the amount, at the end of 2 years, 10 months, may be £627 - 18 - 6? *Answ.* £564 - 0 - 1.

Rule VII. *To find the time in which, at a given rate per cent. per annum, a given principal would produce a given interest:*

$$\left.\begin{array}{l}\text{As the principal : £100}\\ \quad\text{the rate} \quad : \text{the interest}\end{array}\right\} :: 1 \text{ year : the time required.}$$

Exer. 61. In what time will £460 amount to £500, at $4\frac{1}{2}$ per cent. per annum? *Answ.* 1 year, 340 days.

62. How long must £2000 be lent, at simple interest, at $3\frac{1}{2}$ per cent. per annum, to amount to £2280? *Answ.* 4 years.

63. How long must £887 - 5 - 0 be lent, at $5\frac{1}{4}$ per cent. per annum, simple interest, to gain £120? *Answ.* 2 years, 210 days.

Rule VIII. *To find at what rate a given principal would gain a given interest in a given time:*

$$\left.\begin{array}{l}\text{As the given time : 1 year}\\ \quad\text{the principal} \quad : \text{£100}\end{array}\right\} :: \text{the interest : the rate.}$$

Exer. 64. If a merchant, with a capital of £5000, gain £2000 in $2\frac{3}{4}$ years, at what rate per cent. per annum, simple interest, has he gained? *Answ.* £14 - 10 - 11, nearly.

65. If £1 amount to £1 - 2 - 9 in $3\frac{1}{4}$ years, at simple interest, at what rate per cent. per annum must it have been lent? *Answ.* £4 - 4 - $7\frac{5}{13}$.

It may be proper here to insert a few questions which are of a useful kind, and which are of the *same nature* as those regarding the interest of money.

Exam. 10. The population of Glasgow was 77,385 in the year 1801, and 100,749 in 1811. Required the rate per cent. of the increase during the interval.

By taking the difference of these we find 23,364, the increase of the population. Then, as 77,385 : 23,364 :: 100 : 30·19, the rate required.

Exam. 11. Between 1801 and 1811, the population of Edinburgh increased by $24\frac{3}{4}$ per cent., and in the latter year it was 102,987. What was it in 1801?

As $124\frac{3}{4}$: 100 :: 102,987 : 82,555, nearly.

Exer. 66. The population of Glasgow was 147,043, in 1821; 202,426, in 1831; and 282,134, in 1841. Required the rates per cent. of increase during each interval of 10 years, between 1811 and 1841. (See Example 10.) *Answ.* 45·95 nearly, 37·664, and 39·376.

67. The population of Edinburgh was 138,235, in 1821; 162,403 in 1831; and 164,363, in 1841. Required, as in the last exercise (See Example 11.) *Answ.* 34·226 nearly, 17·483, and 1·207 nearly.

68. The population of Great Britain was 14,391,631, in 1821; 16,262,301, in 1831; and 18,526,567, in 1841. Required the rates per cent. of the increases between 1821 and 1831, and between 1831 and 1841. *Answ.* 12·998, and 13·923.

69. The population of Ireland was 6,801,827, in 1821; 7,767,401, in 1831; and 8,205,382, in 1841. Required, as in the last exercise. *Answ.* 14·196, nearly; and 5·639, nearly.

DISCOUNT.

DISCOUNT is an abatement made for advancing money, before it becomes due.

The money which is received as the full payment of a debt or bill, due some time after, is called its PRESENT WORTH.

RULE I. *To find the present worth of a bill or debt:* (1.) Find the interest of the debt, at the given rate, and for the given time: (2.) Consider this interest as discount, and subtract it from the debt to find the present worth.

Exam. 1. Required the present worth of a bill of £170, due at the end of 3 months, at 5 per cent. per annum.

Here, by the method already explained in interest, the discount is readily found to be £2 - 2 - 6; and this being taken from £170, the remainder, £167 - 17 - 6, is the present worth.

Exam. 2. What is the present worth of a bill of £39 - 5, due on the 1st September, but paid on the 3rd July preceding, discount being allowed at 5 per cent. per annum?

The time here is 60 days, for which the interest of £39 - 5 is found to be 6*s.* 5½*d.*; and, by subtracting this from £39 - 5, we have remaining £38 - 18 - 6½, the present worth.

In Great Britain and Ireland three days, called DAYS OF GRACE, *are allowed after the time a bill is nominally due, before it is legally due.* Thus, suppose a bill were drawn on the 8th of April, at 4 months, it would be due not on the 8th, but on the 11th, of August.

It may be remarked, that if, without the days of grace, a bill should appear to be due on the thirty-first of a month which contains only thirty days, the last day of that month is to be taken, and not the first of the next; and, consequently, the third of the next month will be the day on which, by the addition of the days of grace, the bill will be really due. Thus, a bill drawn on the thirty-

first of August, at three months, would be due on the third of December. In like manner, a bill which, without the addition of the days of grace, would be due on the twenty-ninth, thirtieth, or thirty-first of February, if that month contained so many days, would be really due on the third of March. It may be farther remarked, that bills which fall due on Sunday are paid, in Great Britain and Ireland, on Saturday.

Exam. 3. Required the present worth of a bill of £77, drawn 8th March, at 6 months, and discounted 3rd June, at 5 per cent. per annum.

By counting forward 6 months and 3 days from the 8th of March, we find this bill to be due on the 11th September. The number of days, from the 3rd of June till this date, is 100; and the interest of £77, for 100 days, at 5 per cent. per annum, is found, by any of the methods formerly explained, to be £1 - 1 - $1\frac{1}{4}$, and, consequently, the present worth is £75 - 18 - $10\frac{3}{4}$.

Exercises. Required the present worths of the following bills, at the given rates per cent. per annum:—

	Exercises.					*Answers.*		
	£	*s.*	*d.*	*Drawn.*	*Discounted.*	£	*s.*	*d.*
1.	416	3	4,	Mar. 1, at 7 months	June 9, at 4 ...	410	16	$7\frac{1}{4}$
2.	56	0	0,	Sept. 5, — 5 ———	Nov. 12, — $4\frac{1}{2}$...	55	7	$10\frac{1}{4}$
3.	218	11	8,	Aug. 14, — 4 ———	Oct. 3, — 4 ...	216	15	$8\frac{3}{4}$
4.	607	3	4,	May 22, — 5 ———	July 10, — $5\frac{1}{2}$...	597	7	$6\frac{1}{2}$
5.	895	12	0,	Jan. 5, — 11 ———	May 9, — 5 ...	869	9	$4\frac{1}{2}$
6.	16	10	0,	Sept. 25, — 5 ———	Nov. 30, — $6\frac{1}{4}$...	16	4	11
7.	588	12	8,	Mar. 6, — 6 ———	June 11, — 6 ...	579	18	6
8.	486	18	8,	Mar. 25, — 10 ———	June 19, — 5 ...	472	1	2
9.	875	5	8,	Feb. 25, — 7 ———	June 4, — 5 ...	861	7	6
10.	388	2	6,	Dec. 8, — 6 ———	Mar. 25, — 6 ...	383	2	$11\frac{3}{4}$
11.	1000	0	0,	Feb. 16, — 11 ———	Sept. 12, — $5\frac{1}{2}$...	980	11	$2\frac{3}{4}$
12.	568	12	9,	Apr. 27, — 7 ———	June 3, — 5 ...	554	12	4
13.	447	12	6,	June 23, — 6 ———	July 8, — $5\frac{3}{4}$...	435	11	4
14.	22	10	0,	Mar. 31, — 7 ———	May 8, — $6\frac{1}{2}$...	21	15	8
15.	649	13	4,	Nov. 9, — 9 ———	Apr. 19, — $5\frac{1}{2}$...	688	8	2

Exer. 16. Required the discount on £24 - 16 - 0, for one year, at $5\frac{1}{2}$ per cent. per annum. *Answ.* £1 - 7 - $3\frac{1}{4}$.

17. What is the discount on £549, for 32 days, at 5 per cent. per annum? *Answ.* £2 - 8 - $1\frac{1}{2}$.

18. What is the present worth of £970 - 18 - 4, due at the end of 19 months, at $4\frac{3}{4}$ per cent. per annum? *Answ.* £897 - 17 - 11.

The rule which has been given above for the calculation of discount, is that which is always employed in actual practice. It is founded, however, on a principle radically false; and always *gives the discount too large, and consequently the present worth too small, by the interest of the true discount.* This will appear manifest if we consider, that *the true present worth of any debt is such a sum as would, if lent at interest at the assigned rate, amount to that debt at the time at which it would have been due*: and, consequently, the discount, or the difference between the present worth and the debt should be, not the interest of the debt, but the interest of the present worth; and, therefore, the interest of the debt will exceed the true discount, that is, the interest of the present worth, by the interest of that discount. The *true* present worth will be found by the following analogy:—

RULE II. *As the amount of £100, for the given time, and at the proposed rate, is to £100, so is the debt to its true present worth*; and the present worth being subtracted from the debt, the remainder is the discount.

The *reason of the rule* will be evident from the consideration, that £100 is the present worth of its amount regarded as a debt; and, consequently, the analogy given above will become simply this:—as the amount of £100, considered as a debt, is to £100, the present worth of that debt, so is any other debt to its present worth. It is obvious also, that, for the first two terms of the analogy, we might use the amount of *any sum whatever*, and that sum itself; but it is generally more simple and easy to employ £100 and its amount.

To exemplify this rule, let it be required to find the true present worth of £200 due at the end of a year, at 5 per cent. per annum. In this case the amount of £100 being £105, we have, by the rule, this analogy: as £105 : £100 :: £200 : £190 - 9 - $6\frac{1}{4}$, the present worth required. By the common rule, the result would have been £190, with an error, therefore, of 9*s.* $6\frac{1}{4}$*d.*

Again, let it be required to find the true present worth of £463, for 7 months, at 5 per cent. per annum. Here the amount of £100 is £102 - 18 - 4; and, therefore, as £102 - 18 - 4 : £100 :: £463 : £449 - 17 - $6\frac{3}{4}$, nearly. By the common method, the result would be £449 - 9 - 11, and the error 7*s.* $7\frac{3}{4}$*d.*

This question, and all similar ones, may be very easily wrought by the following rule:—

RULE III. Multiply the number of months by the rate, and add the product to 1200; then as the sum is to 1200, so is the debt to its true present worth.

Thus, in the preceding example, we should have this analogy: as 1235 : 1200 :: £493 : £449 - 17 - $6\frac{3}{4}$. The reason of this rule may be thus shown: as 12 months : 7 months :: £5 : £$\frac{35}{12}$, the interest of £100 for 7 months. Then as £$100\frac{35}{12}$: £100, or by reduction of both to twelfths, as 1235 : 1200 :: &c.

As another example, let it be required to find the true present worth of £512, due on the 19th of September, but paid on the 8th of May preceding.

Here, the number of days being 134, we have this analogy: as 365 days : 134 days :: £5 : £1 - 16 - $8\frac{1}{2}$, the interest of £100 for 134 days; and then, as £101 - 16 - $8\frac{1}{2}$: £100 :: £512 : £502 - 15 - $5\frac{1}{4}$, *answer*.

This question, and all others in which it is required to find the true present worth of a sum for a given number of days, may be wrought more easily and more accurately by the following rule:—

RULE IV. Multiply the days by the rate, and add the product to 36,500 ($= 365 \times 100$); then, as this sum is to 36,500, so is the debt to its true present worth.

Thus, in the present example, $134 \times 5 = 670$, and $36{,}500 + 670 = 37{,}170$: then as 37,170 : 36,500 :: £512 : £502 - 15 - 5, the present worth.

This rule depends on the same principle as the last; and the reason of it may be thus illustrated. As 365 days : 134 days :: £5 : £$\frac{670}{365}$, the interest of £100 for 134 days, at 5 per cent. per annum. Then, as £$100\frac{670}{365}$: £100; or by reduction to three-hundred-and-sixty-fifths, as 37,160 is to 36,500, so is the debt to its present worth. Both these rules are, in reality, the same as Rule VI. in Interest.*

The following considerations will be useful in showing the falsity of the common method of discount.

If a person have a bill for £100, payable at the end of a year, at 5 per cent., he will receive, according to the common method of discount, only £95 for it; and were he to lend this sum for a year, at the same rate, instead of £100, to which it obviously should amount, he would receive only £99 - 15. The true present worth is £95 - 4 - $9\frac{1}{7}$, and, consequently, the error 4*s*. $9\frac{1}{7}$*d*. Again, had the same bill been payable at the end of two years, the present worth, by the common method, would have been £90, while it should be £90 - 18 - $2\frac{2}{11}$. The error is consequently 18*s*. $2\frac{2}{11}$*d*., and £90, instead of amounting to £100 at the end of two years, would amount to no more than £99. Had the time been four years, the present worths would have been £80, and £83 - 6 - 8, and the error £3 - 6 - 8. The amount also of the present worth, £80, would be £96, and, consequently, £4 less than it should be. If the time had been 10 years,

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* Unless the time be great, the *true present worth* may be readily derived by approximation from that found by the common method, by finding the interest of the interest first found, and *adding* it to the present worth found by the common method; then, by finding the interest of this last interest, and *subtracting* it from the approximate present worth; and so on, by adding and subtracting alternately the interest of the last interest, till the correction becomes so small, that it would be unnecessary to carry the operation farther. When the time is very short, the true result will often be obtained as easily in this way, as by the principles above explained.

As an example, let it be required to find the present worth of a bill of £140, due at the end of 6 months, at 4 per cent. per annum.

Here the interest of £140 is £2 - 16; that of £2 - 16 is 1*s*. $1\frac{1}{2}$*d*.; and that of 1*s*. $1\frac{1}{2}$*d*. is a farthing. Then, by subtracting £2 - 16 from £140; by adding 1*s*. $1\frac{1}{2}$*d* to the remainder; and, lastly, by subtracting a farthing from that result, we find the present worth to be £137 - 5 - $1\frac{1}{4}$, which is correct.

the present worths would have been £50, and £66 - 13 - 4, where the error is £16 - 13 - 4; and the amount of the present worth, £50, would be £75, instead of £100. Finally, were the time 20 years, the present worth, according to the common method, would be *nothing*, while it should be £50; and were the time greater than 20 years, the present worth would be unassignable, as it would appear to be less than nothing; or, if any meaning could be attached to the result of the operation, it would be, that the person who held the bill, instead of receiving anything for it, would be required *to pay* something to get it off his hands.

From these examples, it will appear how very erroneous the common method of computing discount is, when the time is long.* In every case the discounter of the bill has a greater rate of interest for his money than the nominal one; and the longer the time, the greater is this rate. Thus, to recur to the last series of examples, since, by paying £95 at present, the discounter will be entitled to £100 at the end of a year, he obviously gains £5 on £95; and, therefore, £95 : £5 :: £100 : £5 - 5 - $3\frac{3}{19}$, his gain per cent. In like manner, if the time were two years, the gain per cent. would be found to be £5 - 11 - $1\frac{1}{9}$; if four years, £6 - 5; if 10 years, 10 per cent.; and if 19 years, cent. per cent.

It is true, indeed, that when the time is short, as it generally is in real business, the results found by the two methods are pretty nearly the same; and, therefore, the common method, the computation for which is so easy, may be employed without much error. Still, however, the principle is false, as it gives profits to the discounter which are not proportional to the times. It may be said, that those who keep money for the purpose of discounting are entitled to more than the simple common rate. This may be true; but, if the discounter is to have a greater rate, it should be a fixed one, not depending on the time the bill has to run.

Should the learner wish to work discount in the correct method, the exercises at the beginning of this article will serve his purpose as well as any others; and the following are their answers by that method:—

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* It might be shown algebraically, that the error in the common method of calculating the discount or present worth of a given debt, at a given rate, is nearly proportional to the square of the time, when the time is small, or, more properly, when the discount is small compared with the debt. Thus, at 5 per cent. per annum, the error on a bill of £1000, for 2 months, is nearly 1*s.* $4\frac{1}{2}$*d.*, while, for 4 months, it is nearly 5*s.* $5\frac{1}{2}$*d.*, or very nearly 4 times 1*s.* $4\frac{1}{2}$*d.*

It might also be shown, that the error in the sums to which the present worth of a given sum, found by the common method, would amount at the given rate, would be exactly proportional to the square of the times. Thus, in the examples in the text, in last page, it appeared that in case of a bill of £100, payable in a year, at 5 per cent. per annum, the amount would be £99 - 15, while, if it were payable in two years, the amount would be only £99, the error being in the one case 5*s.*, and in the other £1, or 4 times 5*s.*

True Answers.

	£	s.	d.		£	s.	d.		£	s.	d.
Exer. 1.	410	17	11½	Exer. 7.	580	1	0½	Exer. 13.	435	17	8
2.	55	7	11¾	8.	472	9	11¾	14.	21	16	1¼
3.	216	16	0¼	9.	861	11	10¼	15.	638	12	0
4.	597	10	7¾	10.	383	4	2¾	16.	1	5	10¼
5.	870	4	2¼	11.	980	18	7¾	17.	2	7	11
6.	16	5	0	12.	554	19	1	18.	903	0	0¾

COMMISSION, INSURANCE, &c.

COMMISSION is the sum which a merchant charges for buying or selling goods for another.

BROKERAGE is a smaller allowance of the same nature, paid usually for negotiating bills, or transacting other money concerns.

INSURANCE, or ASSURANCE, is a contract by which one party, on being paid a certain sum, or PREMIUM, by another, on account of property that is exposed to risk, engages, in case of loss, to pay to the owner of the property the amount of loss, if it do not exceed the sum insured on the property.

RULE I. *To compute the commission, brokerage, insurance, or any other allowance on a given sum, at a given rate per cent.:* Multiply the sum by the rate per cent., and divide the product by 100; or, as £100 is to the rate per cent., so is the given sum to the required allowance.

The work is performed as in the case of simple interest for a year at a given rate per cent.

RULE II. *To find how much must be insured on property worth a given sum, so that, in case of loss, both the value of the property and the premium may be repaid:* (1.) Subtract the rate from £100. (2.) As the remainder is to £100, so is the value of the property to the sum to be insured.

The work is performed by the rules of Simple Proportion.

PROFIT AND LOSS.

That branch of arithmetic which treats of the gains or losses on mercantile transactions, is called PROFIT AND LOSS.

The work is performed by Simple Proportions, stated according to the following rules:—

RULE I. *From the prime cost and the selling price, to find the gain or loss per cent.* As the prime cost is to the gain or loss on that cost, so is £100 to the gain or loss per cent.

RULE II. *To find how a commodity must be sold to gain or lose a certain rate per cent.:* As £100 is to the gain or loss per cent., so is the prime cost to the gain or loss on that cost; and from this and the prime cost, the selling price will be found by addition or subtraction.

It should be particularly remarked, that, *by the gain or loss per cent. is to be understood the sum that would be gained or lost at the given prices,* not on a hundred pounds' worth sold, but *on a hundred pounds laid out in prime cost, and in charges, if there be any.*

RULE III. *From the gain per cent., and the selling price, to find the first cost:* As £100, together with the gain per cent., or diminished by the loss per cent., is to £100, so is the selling price to the prime cost.

EXCHANGE.

The object of EXCHANGE is to find how much of the money of one country is equivalent to a given sum of the money of another.

This amount depends partly on the mint regulations in the two countries, and partly on the course of trade. So far as the exchange depends on the mint regulations, it generally continues the same for a number of years, whilst so far as it depends on the course of trade, it fluctuates from day to day. The first, or comparatively constant part of the exchange, is called the PAR OF

EXCHANGE, whilst the whole ever-varying amount is called the COURSE OF EXCHANGE.

The PAR OF EXCHANGE between two countries is the number of standard coins of one country which is equal in value to one of the standard coins of the other country, if the values of gold and silver are in an assumed constant proportion, and if the value of each metal be the same in both countries.

In some countries gold coins are the standard of value, as in British currency since 1816, in Germany and the Scandinavian States since 1870, while in some, as in British India, silver coins are the standard. The weight of these coins is fixed by the mint regulations of the respective countries, as also their fineness, or proportion of pure gold or silver to alloy in them. Between two countries which both have a gold standard, as Great Britain and Germany, on the assumption that gold is of the same value in both countries, it is easy, by compound proportion, to ascertain how much of the standard coin of account of one country is equal to one of the standard of the coin of account of the other country. As, for instance, how many marks (reichsmarks) and pfennige are equal to £1 or British sovereign? The amount so ascertained is the par of exchange between the countries.

If the par of exchange has to be calculated between two countries where one has a gold standard, like Great Britain, and the other, like India, has a silver standard, it is necessary to take into account, besides the weight and fineness of the respective coins which form the standard of account, the proportion between the values of equal weights of gold and silver. This proportion, though varying considerably after the continuous discovery of very fertile mines, and varying slightly from day to day, is for long periods usually very constant. In the mints of various countries some proportion is generally assumed as a guide for the mint arrangements. The published par of exchange is sometimes calculated at the proportion so assumed, and sometimes at the current price of silver in the market where a gold standard is used. The information as to mint regulations necessary to calculate the par of exchange is published in books on coins and exchanges called *Cambists*. As the same books usually give the par of exchange for some stated proportions of the precious metals, merchants may do a great deal of business with foreign countries without having to perform the calculation of the par of exchange.

If the value of either of the precious metals should undergo a temporary change, so as to alter the proportion on which the usual par is calculated, it is easy for bullion dealers and others interested to compute the par at the altered proportion, for the new par is to the par in books in the exact proportion of the observed ratio between the values of equal weights of gold and silver to the ratio assumed in the par in books. For the more advanced student *the method of calculating the par of exchange is given immediately after the*

explanation of the chain rule in the Arbitration of Exchanges, page 248.

The value of silver has fallen so rapidly since 1872, and the laws of currency in different countries are in so unsettled a condition with a number of States adopting a double standard of both gold and silver, like France and the other States in the Latin Monetary Union, and others, like Russia and Austria, having unconvertible paper currency, that, instead of the table of pars of exchange in previous editions, students are referred to the tables of foreign money in "Bradshaw's Continental Railway Guide," and "Browne's Merchant's Handbook," and the following rule for calculating at any time the proportion in value of equal weights of fine silver and fine gold.

Proportion in Value on Equal Weights of Fine Silver and Fine Gold.

It appears from inscriptions at Karnac that in B.C. 1600 the proportion of the value of equal weights of gold and silver was as 13·33 to 1. In B.C. 400 it was, according to Xenophon, the same in Asia. In B.C. 326 it was 11·50 in Greece. In Rome the proportion was exceptionally high in B.C. 218 (17·14), and exceptionally low when gold came from Aquileia in B.C. 100 (8·00). At the Christian era it was in Rome 12·0. In A.D. 864 in France it was 12. In 1260 in Italy it was 10·5. In 1403 in Germany it was 12·8, and in 1500 it was 10·5. In England in 1604 it was 12·10; in 1619, 13·35; and in 1670, 14·50: in 1851, 15·46; in 1859, 15·21; in 1874, 16·15; and in 1880, 17·98.

The mode in which this proportion is calculated from the price of standard silver in bars in the London market, as quoted in the newspapers, is as follows:—As British standard for gold coins has 11 parts of fine gold to 1 part of alloy, under statute 56 Geo. III. c. 68, continued by 33 Vic. c. 10, the value of an ounce of fine gold is $\frac{12}{11}$ times the value of an ounce of standard gold, i.e. = 1019·45 pence. As British standard silver has 37 parts of fine silver to three of alloy, an ounce of fine silver is worth $1\frac{3}{37}$ times, or $1\cdot\dot{0}8\dot{1}$ times, an ounce of standard silver. If the quoted value in pence of an ounce of standard silver be multiplied by this numeric, the result gives very nearly the value of an ounce of fine silver. If the above statutably fixed value of an ounce of fine gold—1019·45 pence—be divided by the variable value of an ounce of standard silver, viz. (quoted price of an ounce in pence, which we will call) x, multiplied by the above numeric—1·081—we get the ratio of gold to silver $\frac{943}{x}$, very nearly.*

Exer. 1. Required the proportion of value of silver to gold if the price of bar silver in the newspapers be quoted for the London market at $52\frac{7}{16}$ pence. *Answ.* 17·98, nearly.

In countries that have a double standard of value for both gold

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* *Report on Depreciation of Silver*, Par. pap. 1876, No. 338, Appendix, page 33.

and silver, a certain proportion has to be fixed by law. In France, Belgium, Switzerland, and Greece (the countries in the Latin Monetary Union) this was, up to 1876, if not subsequently, fixed at $15\frac{1}{2}$.

Exer. 2. What is the price per ounce troy of standard silver in the London market, which corresponds to the above Latin Monetary Union proportion of value of silver to gold of $15\frac{1}{2}$? *Answ.* $60\frac{3}{4}$ pence, nearly.

If silver falls in value below the proportion fixed in countries with a double standard, the tendency is to make all payments in silver and to export gold coins. To obviate this, in the countries in the Latin Union, the amount of silver that can be coined in each State in the Union is limited, and so an artificial value given to the silver coins by limitation in supply.

In the British currency silver coins are not legal tender above 40 shillings, and are only tokens. A pound troy weight is coined into 66 shillings, so that silver coins are issued at the mint at 5 shillings and 6 pence an ounce; if the price of bar or standard silver should rise above this, there would be a profit in melting silver coins.

Exer. 3. What proportion between the value of silver and gold does the mint rate at which British silver coins are issued represent, viz. 5 shillings and 6 pence per ounce? *Answ.* 14·29, nearly.

COURSE OF EXCHANGE.

The COURSE OF EXCHANGE at any particular time is the sum of the money of one country, which at that time is given for a fixed sum of the money of another. This is seldom at par, but is continually varying according to the circumstances of trade and the market for loans in different countries.* When the student has learned the

* The variations in the course of exchange depend on several circumstances, which, as well as several other questions connected with exchanges, it does not accord with the plan of this publication to explain, such discussions belonging properly to political economy. Suffice it to say that these variations are of two kinds, the first, arising from the state of trade and the market for loans, called the *real exchange*; the second, arising from the currency of either country not being according to the mint regulations upon which the par in use was calculated, is called the *nominal exchange*. Whenever, in consequence of a deficient crop in any country, such quantities of corn and other provisions have to be imported as will exceed the value of the exports to the places from which these articles were brought; or when, in consequence of war, subsidies have to be paid to foreign powers, or remittances have to be made abroad for the payment of forces on foreign service; or, in a word, when any cause makes the imports from a country greater than the exports to it: in all these cases the bills of the foreign State must increase in value, as more of them will be required to be sent abroad to pay the balance; and thus the money of that country will increase in value, and will continue above par till coin or bullion is remitted to make up the deficiency; or till, by a greater exportation, or a smaller importation, or both, on the part of the debtor country, the equilibrium of trade is restored. It is of consequence to observe that the course of real exchange can never differ *very much* from the par,

general principles of exchange and methods of calculation, in applying them to actual business he should recollect that the mint regulations and laws of different countries are liable to alteration, and he should therefore always consult the latest edition of Tait's "Modern Cambist," or some similar work of authority for the existing pars of exchange, for any given price of silver, or for the information necessary to calculate the par at any other rate.

In foreign exchanges one place always gives another a *fixed sum of money* for *a variable sum*. The former is called the CERTAIN AMOUNT; the latter the VARIABLE AMOUNT.

To understand the quotations of the course of exchange given from day to day in the newspapers, it is necessary to bear in mind that two quite distinct methods of expressing the course of exchange have grown up.

I. With some countries, usually those where exports exceed imports, and bills for balance drawn in British money, the variable amount of foreign currency at the moment exchangeable for the British unit of currency is quoted. Thus, with Germany, 20 marks, 43 pfennige for £1; with France, 25 francs, 22½ centimes for £1; and with Holland, 12 florins and 2 stivers for £1.

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as coin or bullion will be remitted instead of bills whenever the course of exchange is such that the value of coin or bullion, and the expense of remitting and insuring it, would be less than the cost of bills. Besides, when the course of real exchange is against a country, it affords an inducement to merchants to export to the other country, as the bills which they will get in return will be more valuable, in consequence of the money of the foreign country being above par; and thus they can procure a better and surer market for their commodities, as they will be enabled to sell them at a lower price. The nominal exchange may differ to any extent from the par of exchange. It has no tendency to correct itself, and can only be corrected by a reformation of the defective currency, or the adoption of the true par according to the last mint regulations as to par in use.

Illustrations of the nominal exchange will be found in the countries stated in *Bradshaw's Continental Railway Guide* to have a paper currency, as in 1880 Austria, Italy, and Russia.

The course of exchange with Vienna was recently 12 florins, 20 cents for £1, whilst the par is stated at 10 florins, 21 creutzers, or cents. This nominal exchange arises from paper money being a legal tender, and issued to excess. So in Russia, while a silver rouble should be worth 38 pence, a paper rouble is worth only 29 pence. The same cause produced a depreciation of the British and Irish currency between 1793 and 1819, amounting at one time to 25 per cent. A large part of the high prices which prevailed during the French war, commonly called war prices, were really depreciated currency prices. The termination of the depreciation of the British currency, and of the nominal exchange against England, was Sir Robert Peel's celebrated Act for the restoration of cash payments in 1819.

II. With other countries, usually those where imports exceed exports, and bills for balance drawn in foreign money, the variable amount of British currency for the moment exchangeable for the unit of foreign currency is quoted. Thus with British India, 1*s.* 8$\frac{5}{8}$*d.* for 1 rupee; with the British colony of Hong Kong, 3*s.* 10$\frac{1}{4}$*d.* for 1 dollar; with the Empire of China, 5*s.* 3$\frac{1}{2}$*d.* for 1 tael or leang; with Portugal, 53 pence for 1 milree; with Spain, 47$\frac{1}{2}$ pence for 1 peso (5 pesetas).

All the calculations in exchange may be performed by the rule of proportion; and the operation may often be abbreviated by the method of aliquot parts.

General Rule. (1.) Place, as the second term in the analogy, that sum whose value is to be found in the money of another country: (2.) Make that term of the rate which is of the same kind with the second term, the first term of the analogy, and the remaining term of the rate the third term: (3.) Then work the analogy in the usual way.

ARBITRATION OF EXCHANGES.

When the courses of exchange between the first and second, the second and third, the third and fourth, &c., of any number of places, are given; the method of finding the course of exchange between the first place and the last, corresponding to these courses, or of valuing any sum of the money of the first place in that of the last, through the medium of the others, is called ARBITRATION OF EXCHANGES.

As the actual course of exchange between the first place and the last, is almost always, from various circumstances, different from the arbitrated course, this method is of use in enabling a merchant, in one place, to discover whether he should draw and remit directly between his own place and another, or circuitously through other places.

All the operations may be performed by one or more analogies in the rule of proportion.

The method by the rule of proportion is easy and intelligible, when there are only three places concerned, or in what has been termed SIMPLE ARBITRATION. But when more than three places are concerned, or in what has been called COMPOUND ARBITRATION, the following rule, commonly called the CHAIN RULE, is generally preferable. This rule is also applicable in simple arbitration.

CHAIN RULE.

RULE. (1.) Let all the quantities of the same kind be reduced to the same denomination, if they be not so already. (2.) Let a blank* be left for the required quantity, and, to the right of it, place, as *consequent*, the term to which it is to be equivalent: then, below the blank, place, as *antecedent*, the other given term which is of the same kind as the last consequent, and to the right of it, as consequent, the term which is equivalent to it. (3.) Proceed thus, till all the terms are arranged in two columns, then divide the continual product of the consequents by that of the antecedents, and the quotient will be the result required.

The operation may often be abridged by striking out any antecedent and consequent that are equal, or by dividing an antecedent and consequent by a common measure.†

This rule also shows that it is nothing else than simple or compound proportion, exhibited in a convenient form for its purpose. It will be seen, that, in the use of the chain rule, the first antecedent and the last consequent are always of the same kind.

Exam. 1. When exchange between England and America is at £108 for 444 dollars, 44 cents, and between England and Amsterdam at 10 florins, 90 cents, for £1, what is the arbitrated course of exchange between America and Amsterdam?

Here, if we multiply both sides by 100, we turn the cents into dollars, and the centimes into florins; then we have

Dutch	?	= 1 dol. American
American	444 d. 44 c.	= £108 English
English	£1	= 10 fl. 90 c. Dutch.

$$\frac{108 \times 1090}{44444} = \frac{27 \times 1090}{11111} = 2 \text{ fl. } 64\tfrac{3}{4} \text{ cents,}$$

which is the arbitrated course of exchange for 1 dollar United States. This can be converted into its equivalent $37\frac{3}{4}$ cents United States for 1 florin Dutch, as the exchange is often stated.

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* The interrogative mark (?) may properly be placed in the blank.

This rule is expressed in general terms, so as to serve not only for the purposes of exchange, but for the comparison of weights and measures, and for any other uses to which it can be applied.

† The application of logarithms greatly facilitates, in many cases, the operations by the chain rule. In using them in this way, the contraction above mentioned is generally of little use, unless when terms can be rejected.

Exam. 2. If a merchant in England owe a merchant in Portugal £572, whether is it better for the Portuguese merchant to have a direct remittance from London to Lisbon at 68*d.* per milree, or a circuitous one through Amsterdam and Paris, exchange between London and Amsterdam being at 11 fl. $17\frac{1}{2}$ cents Dutch per pound sterling; between Amsterdam and Paris at 140 florins for 300 francs; and between Paris and Lisbon at 460 rees for 3 francs, an expense of $1\frac{1}{2}$ per cent. being incurred in the circuitous course?

Portuguese	? = £572	English
English	£1 = 11 fl. $17\frac{1}{2}$ cents	Dutch
Dutch	140 fl. = 300 fr.	French
French	3 fr. = 460 rees	Portuguese
Portuguese 100 rees	= $98\frac{1}{2}$ rees,	expenses deducted

$$\frac{572 \times 2235 \times 300 \times 460 \times 197}{28000 \times 3 \times 200} = \frac{572 \times 2235 \times 46 \times 197}{28 \times 200}$$

$$= \frac{143 \times 447 \times 23 \times 197}{7 \times 20} = 2068757 \text{ rees} = 2068 \text{ milrees, } 757 \text{ rees,}$$

the sum Portuguese by the circuitous course. Again, as 68*d.* : £572 : : 1 milree : 2018 milrees, 824 rees, the sum Portuguese by direct remittance. Hence the circuitous course will be more advantageous to the Portuguese merchant, as by it he will receive 49 milrees, 933 rees more than by the direct.

Below the columns we place 100 and $98\frac{1}{2}$, and multiply by them, to modify the result according to the loss per cent.; $98\frac{1}{2}$ being equal to $100 - 1\frac{1}{2}$.

Exer. 4. If a merchant in Russia owe a merchant in London 12,000 roubles; and if the course of exchange between London and St. Petersburg be $36\frac{1}{4}$*d.* per rouble; how much more or less profitable is it for the London merchant to draw directly on St. Petersburg, or to draw through Paris, Amsterdam, Hamburg, and Vienna, the course of exchange between London and Paris being at 24 f. 55 c. per pound sterling; between Paris and Amsterdam at $137\frac{1}{2}$ florins Dutch for 300 francs; between Amsterdam and Hamburg at 38 florins for 40 marks Hambro'; between Hamburg and Vienna at 100 marks Hambro' for 89 florins 70 creutzers of Vienna; and between Vienna and St. Petersburg at 185 creutzers per rouble? *Answ.* The direct remittance is better by £10 - 7 - $11\frac{1}{2}$.

PAR OF EXCHANGE.

From the definition of Par of Exchange (see page 243), it appears that to calculate the par between two places having the same metal as standard of currency, is only to ascertain the proportion between the weight of the precious metal in the standard coin of the one country, and the weight of the same metal in the standard coin

of the other country. This is done by the Chain Rule, as in the following example :—

Exam. 3. What is the par of exchange in gold dollars for £1 between the United States and London; 40 lbs. troy of British standard gold being coined into 1869 sovereigns; British standard being 22 out of 24 parts pure; United States standard being 9 out of 10 parts pure, and 10 United States dollars weighing 10 dwt. 18 grains troy?

United States dollars	? = £1
1869 sovereigns	= 40 lbs. troy, British standard gold
1 lb. troy British standard	= 5760 grains British standard
24 grains British standard	= 22 grains pure gold
9 grains pure gold	= 10 grains United States standard
258 grains U.S. standard	= 10 dollars

$$\frac{40 \times 5760 \times 22 \times 10 \times 10}{1869 \times 24 \times 9 \times 258} = \frac{320 \times 11 \times 1000}{1869 \times 3 \times 129} = \frac{5320000}{723303}$$
= 4 dollars 86 cents.

Exer. 5. When 6400 Portuguese rees contained 203 grains of fine gold, what was the par of exchange between London and Lisbon? *Answ.* 67·36 pence per milree.

If the par of exchange is to be calculated between two places where one has a gold standard and the other a silver standard, the weight of silver which is assumed equivalent to a given weight of gold, is an additional term in the question; this is usually expressed in the price of a given weight of silver in the gold standard currency, as an ounce of British standard silver—worth 50 pence.

Exam. 4. What is the par of exchange between London and British India in pence for 1 rupee, British standard silver being in the London market quoted at 50 pence an ounce troy, a rupee weighing 180 grains troy, 11 out of 12 parts fine silver?

pence	? = 1 rupee
1 rupee	= 180 grains Indian standard
180 grains Indian standard	= 165 grains fine silver
222 grains fine silver	= 240 grains British standard
240 grains troy weight	= $\frac{1}{2}$ an ounce
1 ounce British standard silver	= 50 pence

$$\frac{1 \times 180 \times 165 \times 240 \times \frac{1}{2} \times 50}{1 \times 180 \times 222 \times 240 \times 1} = \frac{165 \times 25}{222} = 19 \text{ pence.}$$

Exer. 6. If the Cologne mark of fine silver be equivalent to $27\frac{3}{4}$ marks *banco*, and 60 marks Cologne weight equal to 451 ounces troy, what is the par between London and Hamburgh *banco* in silver at 5 shillings and 2 pence an ounce? *Answ.* 13 marks $3\frac{1}{2}$ schillings,* nearly, for £1.

* 1 mark = 16 schillings.

DIVISION INTO PROPORTIONAL PARTS.

RULE. *To divide a given quantity into parts which shall be proportional to given numbers:* As the sum of the given numbers is to any one of them, so is the entire quantity to be divided, to the part corresponding to the number used as the second term of the analogy.

The operation is proved by adding the several results together: if the sum be equal to the quantity to be divided, the work is right.

When all the parts, except one, have been determined, that one may be found by adding the rest together, and taking the sum from the number to be divided. It is better, however, to find them all by proportion, as the operation can then be proved by adding them together.

FELLOWSHIP.

FELLOWSHIP is the method of determining the respective gains or losses of the partners in a mercantile company.

Fellowship is usually distinguished into two kinds, *simple* and *compound*, or *single* and *double*.

In SIMPLE OR SINGLE FELLOWSHIP, the stocks or sums contributed by the several partners, all continue in trade for the same time.

In COMPOUND OR DOUBLE FELLOWSHIP, the stocks continue in trade for different periods.

SIMPLE FELLOWSHIP, AND BANKRUPTCY.

RULE. As the whole stock is to the whole gain or loss, so is the stock of any partner to his gain or loss.

In the same way, the estate of a bankrupt may be divided among his creditors by this analogy:—As the sum of all the claims on the estate is to its value, so is the claim of any creditor, to his dividend, or share of the estate.

This rule is merely a particular application of the general rule for division into proportional parts, given in the last article.

It is usual to calculate the percentage of profit and loss in the first place, and then multiply each person's share by it, and divide by 100 to find his dividend of

profit or share of loss. In some cases, particularly in bankruptcy, it is usual to reckon first the shillings and pence per £1 that can be paid to each creditor, and then find the share of each by multiplication.

COMPOUND FELLOWSHIP.

RULE. (1.) Let all the times be of the same denomination, and multiply each stock by the time of its continuance in trade: (2.) Then, using the products as stocks, proceed according to the rule for simple fellowship.

Exam. A and B enter into partnership: A contributes £600 for 13 months, and B £800 for 10 months. Required the share of each in a gain of £650.

Here the products are £7800 and £8000, the sum of which is £15800. Then, as £15800 : £650, or, by contracting, as £316 : £13 :: £7800 : £320 - 17 - $8\frac{3}{4}$, A's share, and as £316 : £13 :: £8000 : £329 - 2 - $3\frac{1}{4}$, B's share. The sum of these is £650, which proves the correctness of the operation.

$$\begin{aligned} £600 \times 13 &= £7800 \\ 800 \times 10 &= \underline{\ 8000} \\ \text{Sum} &= 15800 \end{aligned}$$

The same result may be obtained by working otherwise, thus:—we divide 650 with ciphers annexed, by 15800, or 13 with ciphers annexed, by 316, and we find for quotient ·04113924; and multiplying this successively by 7800 and 8000, and cutting eight figures from the products, we obtain £320·886072 and £329·11392, or £320 - 17 - $8\frac{3}{4}$ and £329 - 2 - $3\frac{1}{4}$, the same as before.

The reason for this plan will be evident from the consideration, that a stock of £600 for 13 months, would be equivalent to 13 times £600 for 1 month; and one of £800 for 10 months, to 10 times £800 for 1 month. Hence, if these increased stocks be employed, it is evident, that since the times are then to be regarded as equal, the work will proceed in the same manner as in Simple Fellowship.

Exer. 1. A's stock £280 for 5 months, B's £266 - 13 - 4 for 6 months: whole gain £331 - 12 - 6. *Answ.* A's gain £154 - 15 - 2, B's £176 - 17 - 4.

2. A's stock £170 for 8 months, B's £280 for 6 months: whole gain £250. *Answ.* A's gain £111 - 16 - $10\frac{1}{4}$, B's £138 - 3 - $1\frac{3}{4}$.

3. A's stock £248 - 12 - 6 for 10 months, B's £670 for 3 months, C's £512 - 7 - 6 for 6 months: whole gain £439 - 18 - 8. *Answ.* A's gain £144 - 9 - $7\frac{1}{4}$, B's £116 - 16 - 1, C's £178 - 12 - $11\frac{3}{4}$.

4. C's stock £178 - 6 - 8 for 18 months, D's £237 - 17 - 6 for 12 months, E's £536 - 5 for 10 months: whole gain £370. *Answ.* C's gain £103 - 18 - $9\frac{1}{4}$, D's £92 - 8 - $6\frac{1}{2}$, E's £173 - 12 - $8\frac{1}{4}$.

5. A's stock £485 - 18 - 4 for one year, B's £279 - 10 for 9

months, C's £675 - 11 - 8 for 8 months: whole gain £386 - 15. *Answ.* A's gain £163 - 19 - 11, B's £70 - 14 - 11½, C's £152 - 0 - 1¼.

6. A's stock £576 - 15 for 11 months, B's £365 - 4 - 10½ for 15 months, C's £582 - 6 - 8 for 9 months: whole gain £568 - 15. *Answ.* A's gain £211 - 9 - 1¾, B's £182 - 12 - 1¾, C's £174 - 13 - 8½.

7. M's stock £1038 - 13 - 9 for 5 months, N's £692 - 9 - 2 for 9 months, O's £1384 - 18 - 4 for 6 months: whole gain £686 - 1 - 2. *Answ.* M's gain £180 - 10 - 10, N's £216 - 13, O's £288 - 17 - 4.

INVOLUTION.

A POWER of any number is the product obtained by the continual multiplication of that number, taken a certain number of times as factor.

A number, in relation to any power of it, is called the ROOT of that power.

When the proposed number is used *twice* as factor, the product is called the SECOND POWER, or the SQUARE,* of that number; when *three times*, the THIRD POWER or CUBE; when *four times*, the FOURTH POWER; when *five times*, the FIFTH POWER, &c.

Powers are often denoted by writing after the proposed number, and a little higher, the number which shows how often the proposed number is repeated as factor. This number is called the INDEX, or the EXPONENT, of the power.

Thus, 5×5, or 25, is the second power, or the square, of 5, and may be written 5^2, where 2 is the index; while $7 \times 7 \times 7 \times 7$, or 2401, is the fourth power of 7, and may be written 7^4, where 4 is the index, &c. Also 5 is the second or square root of 25, and 7 is the fourth root of 2401.

The method of finding any assigned power of a given number, or, as it is also expressed, the method of

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* The second power of a number is very improperly called the *square* of the number. The square of a *line*, is a geometrical figure having four sides, each equal to the proposed line, and having its adjacent sides perpendicular to each other; but this is evidently by no means applicable to an abstract number. The mistake has arisen from the circumstance, that the *area* of a square is found *numerically* by multiplying the number expressing the length of the side by itself, which is the same as the process by which the second power of that number is determined. It might be shown in a similar manner, that the third power of a number is with equal impropriety called the *cube* of that number, the cube being, not a power of a number, but a solid body. The terms *square* and *cube*, however, as well as *square* root and *cube* root, have been so long and so generally used in this improper sense, that it would perhaps be vain to attempt to correct the error any farther than by frequently using the legitimate expressions, *second power*, *second root*, &c., of any of them.

raising a number to any proposed power, is called INVOLUTION.

From the preceding definition of a power, we have the following rule for involution.

RULE. *To find any assigned power of a given number; or to raise a given number to any proposed power:* Find the continual product of the given number repeated as factor, as often as there are units in the index of the proposed power.

The process may often be abbreviated by multiplying together powers already found. In this case, the index of the power thus found is equal to the *sum* of the indices of the powers multiplied together.

When the given number is either wholly or partly a decimal, the operation may often be much abbreviated, by the rule for contracting the multiplication of decimals given in pages 194, 195.

Exam. 1. Required the fifth power of 23.

Here, by multiplying 23 by itself, we find 529 for the second power of 23. By multiplying this by 23, we get 12167 for the third power. By proceeding in like manner, we find the fourth power to be 279841, and the fifth to be 6436343.

Multiply	23	= 1st power.
	23	
Multiply	529	= 2d ——
	23	
Multiply ...	12167	= 3d ——
	23	
Multiply	279841	= 4th——
	23	
Answ.	6436343	= 5th——

The answer might also have been found by multiplying the second power, 529, by itself, and the product 279841, which is the fourth power, by 23. The same result would also be obtained by multiplying the third power by the second.

In like manner, to find the twelfth power of a number, we should multiply the number by itself to find the second power; the second power by itself to find the fourth power; the fourth by itself to find the eighth; and lastly, the eighth by the fourth to find the twelfth.

Exam. 2. Required the fifth power of $\frac{3}{8}$.

The fifth power of 3 is 243, while that of 8 is 32768: the answer therefore is $\frac{243}{32768}$. The *reason* of this is evident from the method of multiplying fractions.

Exam. 3. What is the third power of $1\frac{1}{4}$?

This, by reduction to an improper fraction, becomes $\frac{5}{4}$; and by involving the numerator and denominator each to the third power, we find for answer $\frac{125}{64}$, or $1\frac{61}{64}$. Each of the last examples might

have been wrought by reducing the fractions to decimals, and then working by the general rule.

Exam. 4. Required the sixth power 1·12, true to five places of decimals.

By raising this to the third power, in the way already shown, we find 1·404928, and this being multiplied by itself, as in the margin, we find for the sixth power 1·973822.

$$\begin{array}{r l}
1·404928 & = \text{3rd power.} \\
\underline{829404·1} & \\
1404928 & \\
561971 & \\
5620 & \\
1264 & \\
28 & \\
\underline{11} & \\
1·973822 & = \text{6th ——}
\end{array}$$

Exercises. Involve the following numbers to the powers denoted by their respective indices :—

Exer.	*Answ.*	*Exer.*	*Answ.*
1. 678^2	459684	9. $4·367^4$	363·691179
2. 119^3	1685159	10. $1·03^{17}$	1·652848
3. 75^4	31640625	11. $1·035^{18}$	1·857489
4. 86^5	4704270176	12. $1·04^{15}$†	1·800943
5. 9^9	387420489	13. $1·0475^{24}$	3·045767
6. $(\frac{2}{3})^6$*	$\frac{64}{729}$	14. $1·05^{31}$	4·538039
7. $(2\frac{3}{4})^5$	$157\frac{243}{1024}$	15. $1·055^{26}$	4·0231290
8. $(3\frac{2}{7})^4$	$116\frac{1325}{2401}$	16. $1·07^{32}$	8·7152708

EVOLUTION, OR THE EXTRACTION OF ROOTS.

Evolution is the method of finding, or, as it is usually termed, *extracting*, an assigned root of a given number.‡

The Index of a root is a fraction whose denominator

* The brackets enclosing this fraction, which in this use of them constitute the algebraic *vinculum* in one of its forms, denote that the entire fraction, and not its numerator alone, is to be involved to the sixth power. The object in the next exercise, in like manner, is to find the fifth power of $2\frac{3}{4}$.

† Perhaps the easiest mode of working this exercise will be to find the sixteenth power, and divide it by 1·04. In like manner, in the 14th exercise, the thirty-second power may be divided by 1·05.

It is unnecessary to inform the mathematical student that involution, when much accuracy is not required, is greatly facilitated by the use of logarithms; and the same is true regarding evolution.

‡ Evolution may also be defined to be the method of finding a number, the continual product of which, taken a given number of times as factor, will amount to a given number.

denotes the order of the root, and whose numerator is unity.

The root of a number is also expressed by prefixing to the number the sign, $\sqrt{\ }$,* with the number above it which denotes the order of the root. In case of the square of second root, however, the number 2 is omitted.

Thus the fourth root of 10 is denoted by $10^{\frac{1}{4}}$, or $\sqrt[4]{10}$, and means a number whose fourth power is 10; and the second, or square root of 7 is written $7^{\frac{1}{2}}$, or $\sqrt{7}$, and means a number, such, that if it be multiplied by itself, the product will be 7.

To facilitate the extraction of the square and cube roots, it may be proper for the pupil to be familiar with the following tables:—

TABLE I. The *square* of 1 = 1; of 2 = 4; of 3 = 9; of 4 = 16; of 5 = 25; of 6 = 36; of 7 = 49; of 8 = 64; of 9 = 81.

TABLE II. The *cube* of 1 = 1; of 2 = 8; of 3 = 27; of 4 = 64; of 5 = 125; of 6 = 216; of 7 = 343; of 8 = 512; of 9 = 729.

EXTRACTION OF THE SECOND, OR SQUARE ROOT.

RULE I. *To extract the second, or square root of a given number:* (1.) Commencing at the unit figure, cut off, as often as possible, periods of two figures each.† (2.) The first figure of the required root will be the square root of the left-hand period, or of the greatest square contained in it, if the period be not a square itself. (3.) Subtract the square of this figure from the first period; to the remainder annex the next period for a dividend; and, for part of a divisor, double the part of the root already obtained. (4.) Try how often this part of the divisor is contained in the dividend wanting the last figure, and annex the figure thus found to the parts of the root, and of the divisor, already determined. (5.) Then multiply and subtract, as in division; to the remainder bring down the next period, and, adding to

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* This sign is the letter *r*, the initial of the Latin word *radix, a root*, changed in form by rapidity in writing it, and by its appropriation to a particular use.

† In dividing a decimal, or a number consisting of a whole number with a decimal, into periods, the division must also commence at the unit figure, or the decimal point, and must be continued both ways, if there be a whole number; and if there be an odd figure at the end of the decimal, a cipher, or if it be a periodical decimal, the figure that would next arise from its continuation, must be annexed. For example, 417·245 will be divided thus, 4′17′·24′50; 41·33333, &c., thus, 41′·33′33′33, &c.; and ·567 thus, 56′70, &c.

the divisor the figure of the root last found, proceed as before. (6.) Continue the process till all the figures in the given number have been used; and if anything remain, proceed in the same manner to find decimals, adding, to find each figure, two ciphers, or if the given number end in an interminate decimal, the two figures that would next arise from its continuation.

Exam. 1. Required the square root of 365.

```
        3′65(19·1049
        1
    29)265
     9 261
   381)   400
     1    381
 38204)   190000
     4    152816
382089)   3718400
      9   3438801
382098)    279599
```

Here, by placing a separating mark between 3 and 6, the given number is divided into the two periods, 3 and 65. 1, the root of the greatest integral square contained in 3, is then put in the quotient, and its square taken from the first period. To the remainder the next period is brought down, which gives for dividend 265. The first part of the divisor is found by doubling 1, the first part of the quotient. In finding, in the next place, what figure must be annexed to the part of the root already found, though 2 would be contained 13 times in 26, yet we try 9, as we know the next figure cannot be greater than 9. We annex 9, therefore, to the parts of the root and of the divisor already found; and multiplying 29 by 9, and subtracting, we have 4 remaining. Hence, we have for root 19, and for remainder 4. Now, to find decimal figures, a point is put after 19; two ciphers are annexed to the remainder; and 9, the figure last found, is added to 29, the former divisor. We have then for dividend 400, and for part of a divisor 38. This part of the divisor is contained once in 40; and therefore the first figure of the decimal is 1, which is also annexed to 28. In working for the next figure, we have the divisor 382, which, not being contained in 190, a cipher is annexed to the parts of the root and the divisor previously found; and two ciphers are annexed to the dividend to find another figure. The rest of the work proceeds in the same manner, and the root, true to four places of decimals, is found to be 19·1049. The truth of this result is proved by multiplying 19·1049 by itself, and adding the remainder to the product, as the result is exactly 365.

In this, as well as in every other case in extracting roots, in which there is a remainder after all the significant figures have been used, the fractional part of the root would be an interminate decimal, differing from the interminate decimals which we have thus far seen, in its not repeating or circulating, and thus presenting no law by which it can be continued.* In this case, in any practical

* Continued fractions, as will appear hereafter, show the law of continuation of *square* roots. With regard to *other* roots, however, there is no method known which shows the law of their continuation.

application, the extraction is to be carried on, till as many decimal figures are obtained as the degree of accuracy necessary in the result may require.

The principle on which the preceding rule depends, is, that *the square of the sum of two numbers is equal to the sum of the squares of the numbers added to twice their product.* Thus, 34 being $= 30 + 4$, its square is equal to the squares of 30 and of 4, with twice the product of 30 and 4; that is, to $900 + 2 \times 30 \times 4 + 16 = 1156$. Hence, in extracting the second root of 1156, we separate it into two parts, 1100 and 56. Then 1100 contains 900, the square of 30, with the remainder 200: the first part of the root therefore is 30, and the remainder is $200 + 56$, or 256. Now, according to the principle above mentioned, this remainder must be twice the product of 30 and the part of the root still to be found, together with the square of that part. Dividing then 256 by 60, the double of 30, we find for quotient 4, which being added to 60 the sum is 64; and this being multiplied by 4, the product, 256, is evidently twice the product of 30 and 4, together with the square of 4. In the same manner the operation may be illustrated in every case. The rule, however, is best demonstrated by algebra.

RULE II. *When the root is to be extracted to many figures, the process may be much contracted by the following rule:* Find, by Rule I., half, or one more than half, the number of figures required: then to the remainder annex one figure instead of two, and having found the divisor in the usual way, proceed according to the contracted method of dividing decimals.

Thus, suppose it had been required, in the preceding example, to find the answer true to nine places of decimals; these and the two places of whole numbers are eleven figures in all; and, therefore, before commencing the contraction, it is necessary to find six figures. These have been already found to be 19·1049. Taking, therefore, 382098 and 279599, the divisor and remainder already found, and also the quotient 19·1049, the continuation of the preceding work will stand as in the margin, a cipher being annexed to the dividend, according to the rule. The first figure that results from the division is 7, which, in working the operation at full length, must have been annexed to the divisor: we therefore carry 5, for 7 times 7, to 56, the product of 7 and 8. After this the work proceeds exactly as in the contracted mode of dividing decimals; and the root is found to be 19·104973174.

$$\begin{array}{r} 382098)2795990(19\cdot104973174 \\ \cdots\; 2674691 \\ \hline 121299 \\ 114630 \\ \hline 6669 \\ 3821 \\ \hline 2848 \\ 2675 \\ \hline 173 \\ 153 \\ \hline 20 \end{array}$$

RULE III. *To extract the square root of a vulgar fraction:* Reduce it to its simplest form, if it be not so already, and extract the roots of both terms, if they be complete powers: otherwise, find their product, extract its square root, and divide the result by the denominator.

The root may also be found by reducing the fraction to a decimal, and taking the root of the result.

Thus, the second root of $\frac{1}{4}$ is $\frac{1}{2}$. This result may be obtained either by taking the roots of both terms, or by reducing the given fraction to the decimal ·25, the root of which is ·5, or $\frac{1}{2}$, as before.

In like manner, the root of $2\frac{1}{4}$, or $\frac{9}{4}$, is $\frac{3}{2}$, or $1\frac{1}{2}$. This result might also be obtained by extracting the root of 2·25; as this would also be found to be 1·5, or 1.

Again, if it be required to find the second root of $\frac{5}{7}$; let the square root of 35 ($=5 \times 7$), which will be found to be 5·9160798, be divided by the denominator 7, and there will result ·84515425, the root required. The same result would be obtained by extracting the root of ·71428571, &c., the decimal equivalent to the given fraction.

The pupil will sometimes find the following contractions useful: —1. *If the denominator be an exact square, and the numerator not,* divide the square root of the numerator by the square root of the denominator. 2. *If the numerator be an exact square and the denominator not,* divide the product of the square roots of the numerator and denominator by the denominator.

Exercises. Required the square roots of the following numbers:—

Exer.	*Answ.*	*Exer.*	*Answ.*
1. 5	2·236068	14. $1\frac{3}{8}$	1·172603940
2. ·5	·7071068	15. $1\frac{3}{80}$	1·018577439
3. ·7	·8366600	16. 33	5·7445626465
4. ·07	·2645751	17. 333	18·24828759
5. ·06	·2449490	18. 666	25·806975801
6. ·006	·0774597	19. $\frac{5}{9}$	·74535599250
7. 785	28·01785	20. $\frac{4}{11}$	·60302268915
8. 78·5	8·8600226	21. $\frac{2}{3}$	·81649658093
9. 562	23·70653918	22. $1\frac{13}{36}$	$1{\cdot}1\dot{6}$, or $1\frac{1}{6}$
10. $\frac{3}{8}$	·6123724357	23. $11\frac{1}{9}$	$3{\cdot}\dot{3}$, or $3\frac{1}{3}$
11. $13\frac{1}{5}$	3·633180425	24. $\frac{11}{16}$	·82915619759
12. 1728	41·56921938	25. $6\frac{1}{7}$	2·4784787961
13. $\frac{3}{80}$	·19364916731	26. $794\frac{1}{5}$	28·181554251

27. 123456789	11111·1110605555
28. 987654321	31426·968052932
29. $207\frac{25}{36}$	14·411607975672
30. 822650	907·00055126775
31. $34967\frac{2}{11}$	186·99513848809

EXTRACTION OF THE THIRD, OR CUBE ROOT.

RULE I., or *General Rule. To extract the third or cube root of a given number:* (1.) Place in succession, and at moderate intervals, two ciphers and the given number, as the commencements of three columns. (2.) Beginning at the unit figure of the given number, cut off as many periods as possible, of three figures each. (3.) For the first figure of the root, take the root of the greatest cube contained in the left-hand period. (4.) Place this figure in the first column; and, having added it to what stands above it, multiply the sum by the same figure, writing the product in the second column. (5.) Add, in like manner, in the second column, and multiply the sum by the same figure; set the product in the third column, and subtract it from what stands above it. (6.) Perform a process exactly similar in the first and second columns; and, after that, add the figure found for the root, to what stands in the first column. (7.) Annex one cipher in the first column, and two in the second; and in the third, the next period of the given number; or, if there be no figures remaining, annex three ciphers. (8.) To find the next figure of the root, divide the number in the third column by the one in the second. (9.) Place this figure in the first column, and proceed in the manner directed in Nos. (4), (5), and (6). (10.) Then annex ciphers, &c., as in No. (7); and proceed in exactly the same manner as before, continuing the process till nothing remains, or till the root is carried out as far as may be considered necessary.

Care must be taken to insert the decimal point in the root, when the figures in the integral part of the given number have been all employed.

The work may be contracted by cutting one figure from the number in the second column, and two from the one in the first; and then proceeding according to Nos. (4), (5), and (6), except that the new figure of the root is not to be added in the first column.

The work is proved by involving the root to the third power and adding in the remainder, as the sum ought to be equal to the

given number. If the contracted method be employed, the exact remainder is not obtained. In that case, the more nearly the third power of the root agrees with the given number, the more nearly correct is the root.

Exam. 1. Required the third, or cube root of 926859375.

Here, by Table II., page 256, the greatest cube contained in the first period, 926, is 729; the root of which is 9 the first figure of the required root. This is placed under the first cipher; and, going through the form of adding these, we get 9, the product of which by 9 is set in the second column. Then, by addition, we have 81, the product of which by 9 is 729. This is set under the first period, 926, and subtracted from it, and to the remainder 197, the second period, 859, is annexed. Then, commencing at the first column, we add 9; and multiplying the sum, 18 by 9, we set the product, 162, in the second column, and adding it to the number above it, we get 243. We next add 9 in the first column; and, annexing one cipher in that column, and two in the second, we finish all that is preparatory to the finding of the second figure of the root.

0	0	926'859'375(97
9	81	729
9	81	197859
9	162	183673
18	24300	14186375
9	1939	14186375
270	26239	....
7	1988	
277	2822700	
7	14575	
284	2837275	
7		
2910		
5		
291		

To find that figure, we divide the number in the third column by the one in the second. The quotient would appear to be 8; this, however, would be found on trial to be too large, and we therefore take 7, which answers. We add this in the first column, and multiply the sum, 277, by 7, setting the product in the second column. Then, by adding, we get 26239, the product of which by 7 is put in the third column. By taking this from the number above it, we find for remainder 14186, to which the third period, 375, is annexed. We then add 7 in the first column, and multiplying the sum by 7, we obtain 1988, which is added to the number in the second column. The operation preparatory to the finding of the third figure of the root is then completed by adding 7 in the first column, and annexing one cipher in it, and two in the second.

To find the third figure, we divide, as before, the number in the third column by the one in the second. We thus obtain 5, which is added in the first column, and the sum, 2915, being multiplied by 5, and the product being added to the number in the second column, the sum, 2837275, is multiplied by 5; and the product being

exactly equal to the number in the third column, there is no remainder, and the work terminates, the root being 975.

By raising this to the third power, we get the original number, which proves the correctness of the result.

Exam. 2. Required the cube root of 78314·6.

Here, after the first three figures of the root have been obtained, as in the last example, the numbers in the three columns are 1281, 546987, and 460117. Then, for employing the contracted method, we point off one figure in the second column, and two in the first; and by dividing 460117 by 54698, we get 8 for the next figure of the root. By this we multiply 12, the part not cut off in the first column, and to the product, 96, we add 6, that would be carried from the product of 8 into 8, the first of the figures cut off: the sum, 102, we add in the second column, carrying 1 for the figure 7 cut off in that column, and thus obtaining 54801. The product of this by 8 is then set in the third column, and subtracted, leaving the remainder, 21709. In the second column, again, we add 102, as, it is plain, we should, in effect, have done, had we not contracted. Then, cutting off one figure in the second column, and two in the first, we exhaust the latter, and the rest of the work becomes merely an operation in the contracted mode of division of decimals. The answer is found to be 42·783954. This is true throughout, except the last figure, which ought to be 3, as would be found by proceeding farther, before commencing the contracted process.

0	0	78'314'·600'(42·783954
4	16	64
4	16	14314
4	32	10088
8	4800	4226600
4	244	3766483
120	5044	460117
2	248	438408
122	529200	21709
2	8869	16471
124	538069	5238
2	8918	4941
1260	546987	297
7	102·	274
1267	54801	23
7	102	22
1274	54903	1
7	....	
1281		
. .		

Had we wished to obtain the answer true for one figure more, instead of proceeding as above, we might have added a cipher in the third column, retained the second unchanged, and cut one figure from the first column: or, had we wished to have two additional figures true, we should have annexed two ciphers in the third column, and one in the second, leaving the first unchanged. In either case, the work would proceed upon the same principle, any

trifling variations being such as to be readily followed out by the intelligent student.*

RULE II. *To extract the cube root of a vulgar fraction:* (1.) If, when the fraction is in its lowest terms, the numerator and denominator be exact cubes, extract their roots for the numerator and denominator of the answer. (2.) If only the denominator be an exact cube, the answer will be obtained by finding the cube root of the numerator, and dividing it by that of the denominator. (3.) If the denominator be not a cube number, multiply the numerator by the square of the denominator; take the cube root of the product, and divide it by the denominator.

In every case, the vulgar fraction may be reduced to a decimal, the root of which, taken by the general rule, will be the required root.

The cube root of a mixed number is generally best found, by reducing the fractional part to a decimal, if it be not such already, annexing this decimal to the integral part, and then extracting the root by the general rule.

Exercises. Find the cube roots of the following numbers:—

Exer.	*Answ.*	*Exer.*	*Answ.*
1. 123	4·973190	6. 1234567	107·276572
2. 517	8·025957	7. 44·6	3·546323
3. 900	9·654894	8. $\frac{4}{15}$	·64365958974
4. 123456789	497·9338592	9. $\frac{8}{9}$	·9614997135
5. 12345678	231·120418	10. 376	7·217652

Exer. 11. How much does the sum of the cube roots of 50 and 31 exceed the cube root of their sum? *Answ.* 2·49866.

12. How much is the difference of the cube roots of 50 and 31 less than the cube root of their difference? *Answ.* 2·12575.

EXTRACTION OF ROOTS IN GENERAL.

RULE I. *Any root whatever may be extracted* by an extension of the principle on which the rule that has

* The facility, elegance, and simplicity of the method given above, for the extraction of the cube root, will be apparent to all readers, and more especially to those who have learned any other method of effecting the same object. The more advanced pupil may have recourse to a note at the end of the volume for additional information which could not be introduced with propriety here.

been given for extracting the cube root, depends. In this general application of the principle, the given number is to be divided into periods, each consisting of as many figures as correspond to the order of the root, and the same number of columns is to be employed, the last headed by the given number, and the others by ciphers. The work then proceeds exactly as in the extraction of the cube root; and if there be a remainder, a like contraction is admissible with equal advantage.

Exam. 1. Required the fifth root of $95\frac{2}{11}$.

0	0	0	0	95′·18182′(2·4872061,	
2	4	8	16	32	*answer.*
2	4	8	16	6318182	
2	8	24	64	4762624	
4	12	32	800000	1555558	
2	12	48	390656	1418496	
6	24	80000	1190656	137062	
2	16	17664	468224	133119	
8	4000	97664	1658880	3943	
2	416	19392	11424·	3825	
100	4416	117056	177312	118	
4	432	21184	11792	115	
104	4848	138240	189104	3	
4	448	46·	106·		
108	5296	1428	19017		
4	464	46	106		
112	005760	1474	19123		
4	· ·	46	· · · ·		
116		1520			
4		· ·			
0120					

In this example, as the root is of the fifth order, we have five columns, the first four commencing with zeros, and the fifth with the given number, the fraction being reduced to a decimal carried out to five places, the number constituting the period in extracting the fifth root, and the last being made 2, because of the 8 which would follow it. Then, the first figure of the root is 2, since the fifth power of 2 is 32, while that of 3 is too great, being 243. Now, after a process exactly analogous to that employed in extracting the cube root, we find, that, by adding one cipher in the first column, two in the second, &c., the numbers in the several columns which are to be employed in working for the second figure are 100, 4000, 80000, 800000, and 6318182. The next figure is found to be 4; and, after the usual process, the numbers in the several columns, preparatory to the work for the next figure, turn out to be 120,

5760, 138240, 1658880, and 1555558. Then, in commencing the contraction, we cut one figure from the fourth column, two from the third, three from the second, and four from the first (which last is thus exhausted). The rest of the work presents no difficulty.

This method of extracting roots is perfect in principle, and in most of the useful cases of evolution, it gives, perhaps, as easy solutions as the nature of the case will admit. In some instances, however, especially when the roots are of a high order, the process will be easier by means of the following approximate rule, first given by Dr. Hutton, in the tenth of whose Tracts its investigation may be found:—

RULE II. *To extract any root whatever:* (1.) Call the index of the given power n; and find by trial a number nearly equal to the required root, and call it the *assumed root.* (2.) Raise the assumed root to the power whose index is n. (3.) Then, as $n+1$ times this power, added to $n-1$ times the given number, is to $n-1$ times the same power added to $n+1$ times the given number, so is the assumed root to the true root nearly. (4.) The number thus found may be employed as a new *assumed root*, and the operation repeated to find a result still nearer the true root.

Exam. 2. Required the 365th root of 1·06.

Here we may take 1 for the assumed root, the 365th power of which is 1; and n being 365, we have $n+1=366$, and $n-1=364$. The work will then proceed in the following manner, and the answer will be found to be 1·0001596.

$$\begin{array}{rcl} 1 \times 366 = 366 & & 1 \times 364 = 364 \\ 1{\cdot}06 \times 364 = \underline{385{\cdot}84} & & 1{\cdot}06 \times 366 = \underline{387{\cdot}96} \\ \text{As } 751{\cdot}84 & : & 751{\cdot}96 :: 1 : 1{\cdot}0001596. \end{array}$$

In extracting the fourth root, we may either use one of the preceding rules (Rule I. will be preferable), or we may extract the second root of the given number, and the second root of the result. In extracting the sixth root, also, we may either employ one of those rules, or we may extract the third root of the given number, and the second root of the result: and in this way we may proceed in every case in which the index of the root to be extracted is a composite number. When the index is a prime number, however, the root must be found by one of the general rules.

Exam. 3. Find the value of $11^{\frac{2}{3}}$.

This expression means the third root of the second power of 11, and, therefore, by extracting the cube root of 121, we find for the required result 4·946088. The answer might also be obtained by the second rule, by taking 5 for the assumed root, and $\frac{3}{2}$, the reciprocal of $\frac{2}{3}$, for n. The former method is preferable.

Exam. 4. From the data in Example 10, page 235, let it be required to find the annual rate, per cent., of the increase of the population of Glasgow, between the years 1801 and 1811, supposing the increase to be always proportional to the population.

Here, by dividing the greater population by the less, we get 1·301909, of which, because the interval is ten years, the tenth root is to be taken. This is effected most easily by first extracting the square root, which is found to be 1·141017; the fifth root of which (found very easily by Rule I.) is 1·02674. Then, multiplying this by 100, and rejecting 100 from the product, we get 2·674, the annual rate per cent. required. The reason of the process will be understood by those who have studied compound interest.

Exercises. Find the roots of the following numbers signified by their several indices:—

Exer.	*Answ.*	*Exer.*	*Answ.*
1. $987654321^{\frac{1}{7}}$	19·27274	3. $1{\cdot}05^{\frac{1}{12}}$	1·004074
2. $100^{\frac{1}{100}}$	1·047128	4. $9^{\frac{9}{5}}$	52·19591

SERIES.

A SERIES is a succession of quantities, or terms, that depend on one another according to a certain law.

In every series, the first and last terms are called the EXTREMES, and the rest the MEANS.

Writers on arithmetic usually treat of only two kinds of series, *equidifferent series* and *continual proportionals.* These are of more frequent and general use than other kinds of series, and, on this account, claim more particular attention. Quantities in equidifferent series are also said to be in *arithmetical progression*; and continual proportionals are said to be in *geometrical progression.**

* These names for equidifferent quantities and continual proportionals are very improper. Series of both kinds belong equally to arithmetic and geometry. The appellations, *arithmetical progression* and *geometrical progression*, should, therefore, be entirely disused, as tending to impress false ideas on the mind respecting the nature of the quantities. The term *proportion* is applied, if possible, still more improperly to equidifferent quantities, as this term is always expressive not of *equality of differences*, but of *equality of ratios*. The latest and best Continental writers have accordingly rejected these terms, and substituted more appropriate ones, calling them by the names above given, or others of similar import, such as *progressions by differences*, and *progressions by quotients.* With regard to the name, *continual proportionals*, here applied to the second kind of quantities, it may be observed, that, besides its being perfectly expressive of the nature of such quantities, it has long been thus applied in works on geometry; and it is equally applicable in arithmetic.

EQUIDIFFERENT SERIES, OR ARITHMETICAL PROGRESSION.

When three or more numbers or quantities are such, that each of them, after the first, exceeds the one before it, or that each is less than the one before it, by a constant number or quantity, called the COMMON DIFFERENCE, they are said to be EQUIDIFFERENT, or to be in ARITHMETICAL PROGRESSION.

When each term, after the first, is greater than the one before it, the series is said to be an ASCENDING one; otherwise, it is a DESCENDING one.

Thus, 5, 7, 9, 11, 13, is an ascending equidifferent series, in which each term is derived from the one before it, by the addition of the common difference 2; and 20, 17, 14, 11, 8, is a descending one, in which each term is less than the one before it, by the common difference 3.

The following are the most useful rules for the management of quantities of this kind:—

RULE I. *The first term and the common difference being given, to find any other assigned term:* (1.) Multiply the common difference by the number which is equal to the number of terms preceding the required term. (2.) Then, if the series be an ascending one, add the product to the first term; otherwise, subtract it.

Exam. 1. Required the thirty-fifth term of the increasing equidifferent series, whose first term is 7, and common difference 3.

Here, 34 terms precede the required one; wherefore, $34 \times 3 + 7 = 109$ is the term required.

The reason of this operation will be manifest from the consideration, that were the series to be continued to the thirty-fifth term, the first term must be increased by thirty-four additions of the common difference.

Exer. 1. Required the fifty-fourth term of the decreasing equidifferent series whose first term is 100, and common difference $1\frac{1}{4}$. *Answ.* $33\frac{3}{4}$.

2. Given the first term of an increasing series $= 36$, and the common difference $= 3\frac{3}{5}$; to find the hundredth term. *Answ.* $392\frac{2}{5}$.

3. If the first term of a decreasing series be 329, and the common difference $\frac{7}{8}$; what is the ninety-ninth term? *Answ.* $243\frac{1}{4}$.

RULE II. *The extremes and the number of terms being given, to find the sum of the series:* Multiply the sum of the extremes by the number of terms, and take half the product.

Exam. 2. The first term of an equidifferent series is 1, its last term 312, and the number of terms 193. What is its sum?

Here, $1+312=313$, and $313 \times 193=60409$; the half of which is $30204\frac{1}{2}$, the sum of the series.

The reason of this rule will be understood from the following property of equidifferent quantities:—

In any equidifferent series, the sum of the extremes is equal to the sum of any two terms that are equally distant from them, or to double the middle term, if the number of terms be odd.

Thus, in the series, 5, 8, 11, 14, 17, 20, 23, the sum of 5 and 23 is equal to the sum of 8 and 20, or of 11 and 17, and is double of the middle term 14. The reason of this is plain, since 8 and 11 respectively exceed the less extreme by the same quantities by which 20 and 17 are respectively less than the other extreme.

Hence, in this latter series, it is evident, that though each term were made 14, half the sum of the extremes, still the *sum* of the whole would be the same; and consequently the sum of the series will be obtained by multiplying half the sum of the extremes by the number of terms, or, which is the same, by multiplying the sum of the extremes by the number of terms, and taking half the product.

Exer. 4. Given the greater extreme $=1000$, the common difference $=2\frac{1}{6}$, and the number of terms 367; required the sum of the series. *Answ.* $221484\frac{1}{3}$.

In working this exercise, the less extreme will be found by Rule I., and the sum by Rule II.

5. Given the greater extreme $=1$, the common difference $=\frac{1}{100}$, and the number of terms $=51$; required the sum of the series. *Answ.* $38\frac{1}{4}$.

Rule III. *The extremes and the common difference being given, to find the number of terms:* Divide the difference of the extremes by the common difference, and add a unit to the quotient.

The *reasons* of this rule and of the next will be obvious from comparing them with Rule I.

Exer. 6. If the greater extreme be 500, the less 70, and the common difference 10; what is the number of terms? *Answ.* 44.

7. Given the less extreme $=3$, the greater $=579$, and the common difference $=9$, to find the sum of the series. *Answ.* 18915.

Here, let the number of terms be found by this rule, and the sum of the series by Rule II.

8. With the common difference 12, how many equidifferent means can be inserted between the extremes 8 and 1700? *Answ.* 140.

Rule IV. *The extremes and the number of terms being given, to find the common difference:* Take 1 from the

number of terms, and divide the difference of the extremes by the remainder.

Exer. 9. Given the extremes = 3 and 300 respectively, and the number of terms = 10; to find the common difference. *Answ.* 33.

10. What is the common difference of a series consisting of 1001 terms, the extremes being 1 and 100001? *Answ.* 100.

RULE V. *The extremes being given, to find any assigned number of equidifferent means:* Find the common difference by Rule IV., and add it to the less extreme, or subtract it from the greater; the result will be one mean; from it derive another in the same manner: and thus proceed till all are found.

One mean may be found by taking half the sum of the extremes.

Exam. 3. Given the first term = 1, the last = 99, and the number of terms = 8; required the complete series.

By Rule IV., the common difference is found to be 14, by the continual addition of which to 1, the entire series is found to be 1, 15, 29, 43, 57, 71, 85, 99.

Exer. 11. Insert five equidifferent means between 20 and 30. *Answ.* $21\frac{2}{3}$, $23\frac{1}{3}$, 25, $26\frac{2}{3}$, $28\frac{1}{3}$.

12. Required the several terms of a series, the extremes of which are 4 and 49, and the number of terms 6. *Answ.* 4, 13, 22, 31, 40, 49.

RULE VI. *The sum of the series, one extreme, and the number of terms being given, to find the other extreme:* Divide twice the sum of the series by the number of terms, and from the quotient take the given extreme.

The reason of this rule is evident from Rule II.*

Exer. 13. Given the first term of an equidifferent series, consisting of 24 terms, = 1; required the last term, the sum of the series being = 576. *Answ.* 47.

14. If the number of terms be 50, their sum 1275, and the less extreme $3\frac{3}{4}$; what is the greater extreme? *Answ.* $47\frac{1}{4}$.

---

* If the greater extreme be denoted by g, the less by l, the common difference by d, the number of terms by n, and the sum of the series by s: then $g=l+(n-1)d$ and $s=\frac{1}{2}n(g+l)$; from which two equations, any three of these quantities being given, the rest can be found. The full resolution of these equations will, among other results, give the rules contained in the text.

The consideration of such quantities as are usually said to be in *arithmetical proportion*, has been omitted on account of its comparative inutility. These are quantities of such a nature, that the differences of the first and second, of the third and fourth, of the fifth and sixth, &c., are equal. Such are 2, 5; 10, 13; 21, 24, &c. The principal property of *four* such quantities is, that the sum of the extremes is equal to the sum of the means; and, therefore, if there be four such quantities, and from the sum of the means either extreme be taken, the remainder will be the other.

15. Required the sum of the first ten thousand numbers in the natural series, 1, 2, 3, 4, &c. *Answ.* 50,005,000.

16. Required the sum of the first ten thousand odd numbers, 1, 3, 5, 7, &c. *Answ.* 100,000,000.

17. Required the sum of the first ten thousand even numbers, 2, 4, 6, 8, &c. *Answ.* 100,010,000.

18. Required the sum of the first ten thousand numbers that are divisible by 3 (3, 6, 9, 12, &c.) *Answ.* 150,015,000.

19. If a person on a journey travel the first day 30 miles, and each succeeding day a quarter of a mile less than he did the day before, how far will he travel in 30 days? *Answ.* $791\frac{1}{4}$ miles.

20. How many strokes does a common clock strike in 365 days? *Answ.* 56940.

21. If 120 stones be laid in a straight line, each at the distance of a yard and a quarter from the one next it; how far must a person travel, who picks them up singly and places them in a heap at the distance of 6 yards from the end of the line, and in its continuation? *Answ.* 10 miles 7 furl. 27 per. $1\frac{1}{4}$ yd.

22. A body falling by its own weight, if it were not resisted by the air, would descend in the first second of time through a space of 16 feet and 1 inch; in the next second, three times as far; in the third, five times as far; in the fourth, seven times, &c. Through what space would it fall, at the same rate of increase, in a minute? *Answ.* 57900 feet, or nearly 11 miles.

CONTINUAL PROPORTIONALS, OR GEOMETRICAL PROGRESSION.

When three or more numbers, or quantities, are such, that each of them, after the first, is equal to the product of the one immediately preceding it by a fixed number, they are said to be CONTINUAL PROPORTIONALS, or to be in GEOMETRICAL PROGRESSION.

The fixed multiplier is called the RATIO, or the COMMON RATIO, of the series.

The series is said to be an ASCENDING, or a DESCENDING one, according as the ratio is greater or less than unity.

Thus, 3, 6, 12, 24, 48, form an ascending series of continual proportionals, having the ratio 2; and 192, 48, 12, 3, $\frac{3}{4}$, &c., constitute a descending series, having the ratio $\frac{1}{4}$.*

The following are the most useful rules for the management of continual proportionals:—

* It is proper for the learner to know, that *in a series of continual proportionals, the product of the extremes is equal to the product of any two terms equally distant from them; or to the second power of the middle term, if the number of terms be odd.* Thus, in the series, 3, 6, 12, 24, 48, $3\times48=6\times24=12^2$. The reason of this is evident, since the greater extreme exceeds the term next it in the same ratio in which the other extreme is less than the term next it.

RULE I. *The first term and the ratio being given, to find any other proposed term:* Raise the ratio to the power whose index is equal to the number of the terms which precede the required term, and multiply the first term by the result.

Exam. 1. Required the 8th term of the series of continual proportionals, whose first term is 6, and ratio 2.

Here, the 7th power of 2 is found to be 128; which, being multiplied by the first term 6, the product, 768, is the 8th term.

The *reason* of this operation will be manifest, if it be considered, that in finding the successive terms up to the 8th, the first term must be multiplied by 2, the product by 2, that product by 2, and so on, till the 8th term would be found after seven such multiplications: and it is evident, that the same result will be found by a single multiplication by the 7th power of 2. A similar illustration serves in the case of a decreasing series.

Exam. 2. Required the 20th term of the series, whose first term and ratio are each 1·06.

Here, we are to multiply the 19th power of 1·06 by 1·06, or, which is the same, we are to involve 1·06 to the 20th power. This is found, by involution, to be 3·207135.

Exam. 3. Required the 6th term of the series whose first term is 100, and ratio $\frac{2}{3}$.

The 5th power of $\frac{2}{3}$ is $\frac{32}{243}$; the product of which by 100 is $13\frac{41}{243}$, the term required.

Exer. 1. If the first term of a series be 12, and its ratio 3; what is the 18th term? *Answ.* 1549681956.

2. The first term of a series is 1, and the ratio the reciprocal of 1·07;* required the 14th term. *Answ.* ·4149644.

3. Given the first term of a series, 500, and the ratio the reciprocal of 1·04; to find the 14th term. *Answ,* 300·287.

4. Given the first term of a series = 1, and the ratio = 2; to find the 16th term. *Answ.* 32768.

5. Given the first term of a series = 1, and the ratio = 3; required the 16th term. *Answ.* 14348907.

RULE II. *To find the sum of a series of continual proportionals:* Multiply the last extreme by the ratio, and divide the difference between the product and the other extreme, by the difference between the ratio and a unit: Or, (1.) Raise the ratio to the power whose index is equal to the number of terms. (2.) Divide the difference between the result and unity, by the difference between the ratio and unity. (3.) Multiply the quotient by the first term.

* Here the first term must evidently be divided by the 13th power of 1·07.

When the series is a descending one, and the number of terms infinite, divide the greatest term by the difference between unity and the ratio.

Exam. 4. Given the first term of an increasing series = 4, the ratio = 3, and the number of terms = 6; to find the sum of the series.

Here, by Rule I., we find the last term to be 972. Multiplying this by the ratio, we obtain 2916: and dividing 2912, the difference between this and the first term, by 2, the difference between the ratio and 1, we obtain 1456, the required sum.

The *reason* of this operation is best shown by algebra; it may be illustrated, however, in the following manner: let the terms of the series be placed as in the margin; then let

$$\begin{array}{rl} 4+12+36+108+324+972 & = \text{sum} \\ 12+36+108+324+972+2916 & = \text{sum} \times 3 \end{array}$$

each term be multiplied by the ratio, and let the products be removed each one place towards the right hand. If the upper line be then subtracted from the lower, there will remain 2912 = sum × 2; and consequently the sum is equal to $2912 \div 2 = 1456$. Now, 2916 is evidently the product of the ratio and the greater extreme, and 2912 is the difference between this and the less extreme; also the divisor 2 is the difference between the ratio and a unit: and a similar illustration may be given in any other case.

Exam. 5. Required the sum of six terms of the series, $\frac{1}{2}$, $\frac{3}{10}$, $\frac{9}{50}$, &c.

Here, by dividing the second term by the first, or the third by the second, we find the ratio to be $\frac{3}{5}$; the sixth power of which is $\frac{729}{15625}$. Subtracting this and the ratio each from 1, we get the remainders $\frac{14896}{15625}$ and $\frac{2}{5}$. By dividing the former of these by the latter, we obtain $\frac{7448}{3125}$; the product of which by the first term, $\frac{1}{2}$, is $\frac{3724}{3125}$, or $1\frac{599}{3125}$, the sum required.

Exam. 6. Find the sum of an infinite number of terms of the same series.

To effect this, we divide the first term, $\frac{1}{2}$, by $\frac{2}{5}$, the difference between the ratio, $\frac{3}{5}$, and 1, and we get $\frac{5}{4}$, or $1\frac{1}{4}$, the required sum.

This process will be explained by the illustration in the margin, where the sum on the one hand, and all the terms on the other, are each multiplied by the ratio, $\frac{3}{5}$. The latter product is the same as the given series, except its

$$\begin{array}{rl} \text{Sum} = & \frac{1}{2} + \frac{3}{10} + \frac{9}{50} + \frac{27}{250} + \&\text{c.} \\ \text{Sum} \times \frac{3}{5} = & \phantom{\frac{1}{2}} + \frac{3}{10} + \frac{9}{50} + \frac{27}{250} + \&\text{c.} \end{array}$$

first term. From the whole sum, therefore, subtracting $\frac{3}{5}$ of it, we find that $\frac{2}{5}$ of the sum is equal to the first term of the series, $\frac{1}{2}$; that is, the product of the required sum by the difference between the ratio and 1 is equal to the first term, and hence the *reason* of the rule is manifest.

By the sum of an infinite number of terms of a series, is to be understood the amount to which the sum of more and more of the terms continually approaches, and from which, by taking a sufficient number of them, it may be made to differ by as small a quantity as we please. Thus, in the series, $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+$ &c., the sum of an

infinite number of terms of which is 1, if we take the first and second terms, their sum is less than 1 by $\frac{1}{4}$; if three be taken, the sum is less than it by $\frac{1}{8}$; if four, by $\frac{1}{16}$, and so on, the defect being reduced by one half by the addition of each successive term. It is plain, therefore, that the defect, though it cannot be annihilated, may be rendered as small as we please; so that 1 is the limit to which the sum of the terms, as their number is increased, approaches as nearly as may be desired, but which it can never reach.

Exam. 7. Required the sum of the series whose least term is 45, and ratio 1·04, the number of terms being 31.

Here, by involution, we find the thirty-first power of 1·04 to be 3·37313. Then $3·37313-1=2·37313$, and $1·04-1=·04$; also, $2·37313 \div ·04=59·32825$, the product of which by 45, the first term, is 2669·77125, the required sum.

Exam. 8. Required the value of the interminate decimal $·\dot{1}\dot{8}$.

This is the same as $\frac{18}{100}+\frac{18}{10000}+\frac{18}{1000000}+$ &c., continued without limit, where the ratio is evidently $\frac{1}{100}$. Dividing, therefore, the first term, $\frac{18}{100}$, by $1-\frac{1}{100}$, we obtain for the sum of the series, or the value of the decimal, $\frac{18}{99}$, or, in its lowest terms, $\frac{2}{11}$.

Exer. 6. Required the sum of the series whose first and last terms are 100 and 13·16872428 respectively, and its ratio $\frac{2}{3}$. *Answ.* 273·66255144.

7. Given the extremes $=1$ and 18·42015, and the ratio $=1·06$; required the sum of the series. *Answ.* 308·755983.

8. Find the sum of the infinite series whose greatest term is 100, and ratio the reciprocal of 1·04. *Answ.* 2600.

9. Given the first term of a series $=6$; the ratio $=4$; and the number of terms $=8$; to find the sum of the series. *Answ.* 131,070.

10. Given the first term of a series $=1$; the ratio $=\frac{1}{3}$; and the number of terms $=12$; to find the sum of the series. *Answ.* $1\frac{88573}{177147}$, or $1\frac{1}{2}$ nearly.

11. Given the least term of a series $=1$, the ratio $=1\frac{1}{2}$, and the number of terms $=16$: required the sum of the series. *Answ.* 1311·68167114.

12. Given the first term $=12$, the ratio $=\frac{5}{8}$, and the number of terms $=12$; to find the sum of the series. *Answ.* 63·92472.

RULE III. *The extremes and the number of terms being given, to find the ratio:* Divide the second extreme by the first, and extract that root of the quotient whose index is less by one than the number of terms.

Exam. 9. Given the extremes of a series $=3$ and 192, and the number of terms $=7$: required the ratio.

Here, $192 \div 3=64$, the sixth root of which is 2, the ratio.

Exer. 13. Given the extremes $=1$ and 10, and the number of terms $=9$: required the ratio. *Answ.* 1·333521.

RULE IV. *To find any proposed number of mean proportionals between two given numbers:* (1.) Take the

two given numbers as extremes; take also the number of terms in the series greater by two than the required number of means, and find the ratio by Rule III. (2.) Then the product of the ratio and the first extreme will be one of the means: the product of this mean and the ratio will be another, and so on.

When only one mean is required, it is very easily found by extracting the second root of the product of the extremes.

Exam. 10. Find three mean proportionals between 5 and 1280.

Here, the series would consist of 5 terms, and the extremes are 5 and 1280; and hence the ratio is found, by the last rule, to be 4; and by multiplying the first term by this, the product by the same, &c., the means are found to be 20, 80, and 320.

Exam. 11. Find a mean proportional between 5 and 10.

Here, $5 \times 10 = 50$; the square root of which is 7·0710678, &c., the mean required.

Exer. 14. Find two mean proportionals between 1 and 2. *Answ.* 1·259921, and 1·587401.

15. Find a mean proportional between $\frac{1}{10}$ and 100. *Answ.* 3·16227766.

Exer. 16. If a thrasher agree to work 18 days for a farmer, on condition of receiving two grains of wheat for the first day's work, 6 grains for the second, 18 for the third, &c.: what would be the value of all he would be entitled to receive, supposing 7680 grains to fill a pint, and the wheat to be worth 7 shillings a bushel? *Answ.* £275 - 17 - $5\frac{1}{2}$.

17. Suppose a house, having 20 windows, to be sold at the rate of 4*s*. for the first of these windows, 6*s*. for the second, 9*s*. for the third, and so on; the value of each being increased by one half of itself, to find the value of the next; for how much would it be sold? *Answ.* £1329 - 14 - $0\frac{1}{2}$.

18. It is said that an Indian invented the game of chess, and showed it to his sovereign, who was so much pleased with it, that he desired the inventor to ask any reward he chose. The inventor then asked one grain of wheat for the first square of the chess table, two for the second, four for the third, and so on; doubling continually to 64, the number of squares. Now, suppose it had been possible for the prince to pay this reward, what would have been the value of the whole at 12*s*. 6*d*. per cwt., 10,000 grains being supposed to weigh a pound avoirdupois? *Answ.* £10,293,942,005,418 - 5 - $6\frac{1}{4}$.

19. Find the value of the interminate decimal $\cdot 4\dot{6}\dot{3}$. *Answ.* $\frac{51}{110}$.

20. Required the finite value of the decimal $\cdot\dot{5}18\dot{5}$. *Answ.* $\frac{14}{27}$.

21. Find the sum of the infinite series, $\frac{1}{4}, \frac{1}{8}, \frac{1}{16}$, &c. *Answ.* $\frac{1}{2}$.

22. Find the sum of the infinite series, $\frac{1}{3}, \frac{1}{9}, \frac{1}{27}, \frac{1}{81}$, &c. *Answ.* $\frac{1}{2}$

COMPOUND INTEREST.*

The method that naturally presents itself for finding the amount of a sum at compound interest, is to find its amount at simple interest at the end of the first year; then to take this amount as a new principal, and find its amount in like manner, which would be the amount at compound interest at the end of the second year, and the principal for the third year: the amount of which must be found in like manner. Continuing the process, we should thus find the amount at the end of the proposed time. This will be illustrated in the following example.

Exam. 1. Required the amount of £2500 at the end of 4 years, at 6 per cent. per annum, compound interest.

Here, the amount for one year is £2650; the amount of which for one year also is £2809, the amount at compound interest for two years. The amount of this, again, for 1 year, or the amount of the given sum at the end of the third year, is £2977 - 10 - 9½; the amount of which for another year is £3156 - 3 - 10, the amount of £2500 for four years. The amount at simple interest would have been £3100, which is less than the foregoing amount, by £56 - 3 - 10.

When the time is short, this method may be practised without much trouble; but when it is long, the labour becomes very great. In this case, the methods that follow should be employed.

Rule I. *To find the amount of* one pound *sterling for any number of years, at compound interest;* (1.) Divide the amount of £100 for 1 year by 100, and the quotient will be the amount of one pound for 1 year. (2.) This amount involved to the power denoted by the number of years, will be the amount of one pound for that time.

The contracted mode of multiplication of decimals is peculiarly useful in this rule, and in computations in compound interest and annuities in general. So also is the contracted method of dividing decimals. (See pages 195 and 198.)

Exam. 2. Required the amount of one pound sterling for 20 years, at 4½ per cent. per annum, compound interest.

Here, the amount of £100 for one year is £104·5; the hundredth part of which is £1·045, the amount of £1 for a year. The second power of this is £1·092025, the amount of £1 at the end of the second year. The product of this by itself, found by the contracted method of multiplication, is £1·192518, the amount at the end of the fourth year. The square of this, again, is £1·422099, the amount for eight years; the square of which is £2·022366, the

* For the definition of compound interest, see page 224.

amount at the end of the sixteenth year. Finally, the product of this by £1·192518, the amount for four years, is £2·411708, or £2 - 8 - 2¾, nearly, the amount of £1 for 20 years. At simple interest, the amount would have been only £1 - 18.

Had the products here been found at full length, the labour would have been immense. In the last multiplication, one of the factors would have contained 49 figures, the other 13, and the product 61. It should be carefully remarked, however, that the decimal part of the amount found as above, will rarely be true in all its places. (See page 195.) A trifling error in rejecting or over-estimating a figure at the end of a decimal may accumulate, and render the accuracy of the last figure or the last two figures doubtful. Thus, in the preceding result, the last two figures should have been 14 instead of 08; a difference, however, which would occasion only the trifling error of rather more than a penny in the amount of £1000. When great accuracy is required, the amounts should be brought out to a greater number of places, and the last figure or two of the final amounts rejected, or not depended on. The larger the given sum also, and the longer the time, this is the more necessary, as the effect of the error is the more perceptible.

Exam. 3. What is the amount, true to six places of decimals, of £1 for 6 years, payable half-yearly, at 5 per cent. per annum, compound interest?

Here, the payments being *half-yearly*, the amount of £100 for half a year is £102 - 10, or £102·5; and, consequently, that of £1 for the same time is £1·025: the square of this is 1·050625, or 1·0506250, the amount for two half-years, or one year. Multiplying this by 1·050625, by the contracted method, we obtain 1·1038129, the fourth power of £1·025, or the amount of £1 for two years; the third power of which is 1·3448888, the twelfth power of £1·025, or the amount of £1 for six years; or, if only six figures of decimals be retained, 1·344889.

Exercises. Find the amounts of £1 in the following exercises, at the given rates per cent. per annum:—

	Exercises.			*Answers.*
1.	For 10 years	at 10 per cent., &c.	....	2·593742
2.	— 17 ———	6¼ ———	....	2·802799
3.	— 100 ———	3 ———	....	19·218632
4.	— 100 ———	4 ———	....	50·504948
5.	— 100 ———	5 ———	....	131·501258
6.	— 100 ———	6 ———	....	339·302083

RULE II. *To find the amount, or the interest, of* any sum, *at compound interest, for a given time, and at a given rate:* Find the amount of £1 for the given time, by Rule I., and multiply it by the given sum.

If the principal be subtracted from the amount, the remainder will be the *interest*.

Exam. 4. Required the amount of £760 - 14 - 4 for 12 years, at 5 per cent. per annum, compound interest.

By Rule I., the amount of £1 is found to be £1·795856. This being multiplied by 760, and aliquot parts being taken for 14*s*. 4*d*., as in the margin, the result is £1366 - 2 - 9, the amount; and the principal being subtracted from this, there remains, for the interest, £605 - 8 - 5. The simple interest would have been £456 - 8 - 7¼. The same result would have been obtained by using, instead of 14*s*. 4*d*., the equivalent decimal, and the answer would in that case have been found by multiplication alone.

```
     1·795856
         760 - 14 - 4
  ----------------------
   107751360
  12570992
        897928 for 10s.
        359171 —  4s.
         29931 —  4d.
  ------------
£1366·137590, or
£1366 - 2 - 9, the amount
```

Exercises. Find the amounts of the following sums, at the given rates per cent. per annum:—

	Exercises. £	s.	d.				*Answers.* £	s.	d.
7.	251	16	6	for 9 years, at	5	per cent., &c. ...	390	13	3
8.	212	0	0	— 15 ———	4	——————— ...	381	16	0
9.	213	13	4	— 14 ———	5½	——————— ...	452	2	9¼
10.	463	10	10	— 12 ———	6	——————— ...	932	14	8¾
11.	295	12	6	— 17 ———	4½	——————— ...	624	15	4
12.	495	7	6	— 13 ———	3½	——————— ...	774	14	10¾
13.	649	13	6	— 16 ———	5	——————— ...	1418	3	2¼
14.	582	7	6	— 5 ———	5⅓	——————— ...	761	2	9¾

Exer. 15. If a boy, 12 years old, have a legacy of £1396 - 16 - 8 left to him, how much will he have to receive at the age of 21, the legacy being improved by compound interest, at 5 per cent. per annum? *Answ.* £2166 - 18 - 11¼.

16. Find the amount of £648 from the 6th till the 21st year of a boy's life, at 4¾ per cent. per annum, compound interest. *Answ.* £1299 - 16 - 6½.

17. If a merchant commence trade with a capital of £1200, and each year, after paying all expenses, increase the capital of the former year by a fifth part of itself; how much will he be worth at the end of 30 years? *Answ.* £284,851 - 11 - 6¼.

Rule III. *To find the principal, which, at a given rate, and in a given time, would amount to a given sum; Or, to find the present worth of a sum at compound interest,*

for a given time, and at a given rate: **Divide the given sum by the amount of £1, found by Rule I.**

The present worth of £1 may be found by dividing it by its amount for the given time.

Exam. 5. What sum must be lent at compound interest, at 5 per cent. per annum, at the birth of a child, so that the amount may be £3000 - 6 - 8 at the end of 21 years?

Here, the amount of £1 for 21 years being 2·785962, we have for answer $£3000\cdot\dot{3} \div 2\cdot785962 = £1076\cdot9469 = £1076 - 18 - 11\frac{1}{4}$.

Exercises. Required the present worths of the following sums, or the principals that would produce them, at compound interest, at the given rates per cent. per annum:—

	Exercises. £	s.	d.			*Answers.* £	s.	d.
18.	324	18	6	for 9 years, at	5 per cent., &c.	209	8	$11\frac{3}{4}$
19.	264	11	8	— 12 ———	$4\frac{3}{4}$ ———	151	12	$1\frac{1}{4}$
20.	554	18	4	— 27 ———	4 ———	192	9	1
21.	1000	0	0	— 22 ———	8 ———	183	18	$9\frac{1}{2}$

Exer. 22. A brother is to pay his sister a portion of £4500 at the end of 11 years: how much will discharge the debt at the end of 4 years, compound interest being allowed at $4\frac{1}{2}$ per cent. per annum on the sum he pays? *Answ.* £3306 - 14 - $6\frac{3}{4}$.

23. With what sum must a merchant commence trade, so as to be worth £15,000 at the end of 12 years, if he may be expected to clear annually an eighth of his capital? *Answ.* £3649 - 14 - $7\frac{3}{4}$.

24. Whether is it better to sell a farm for £1000 payable at present, £1000 payable at the end of 5 years, and £1000 payable at the end of 10 years; or to sell it at £3000 payable at the end of 5 years, compound interest being allowed at 4 per cent. per annum? *Answ.* Better at three payments by £31 - 14 - $2\frac{1}{2}$.

The reason of the first rule will appear from the following considerations. The amount of £1 for a year will evidently be a hundredth part of the amount of £100: and as £1 is to its amount for a year, so is any other principal to its amount for the same time. Hence, to take a particular instance, the amount of £1 for a year at 5 per cent. will be 1·05; and by the nature of compound interest, this will be the principal for the second year. Then, as the principal, £1 : £1·05, its amount :: the principal, £1·05 : $£1\cdot05^2$, the amount at the end of the second year, and the principal for the third. Again, as £1 : £1·05, its amount :: the principal, $£1\cdot05^2$: $£1\cdot05^3$, the amount at the end of the third year, and the principal for the fourth. It will thus appear, that the amount of one pound for any number of years, will be equal to £1·05 raised to the power denoted by the number of years. The amount of £1 being thus determined, it is plain that the amount of any other principal will be had by multiplying the amount of £1 by that principal, since the amount will evidently be proportional to the principal; which proves

the second rule. The third rule is evidently the converse of the second, and hence its correctness is evident.*

It may be proper here to observe, that, if interest be payable *yearly*, the amount of £1 at the end of 6 months will be the square root of its amount for a year; its amount for 4 months, or one third of a year, the cube root of the same; for 3 months, the fourth root; for a day, the 365th root, &c. Farther, also, the amount of £1 for 2 days will be the square, for 3 days the cube, &c., of its amount for 1 day; its amount for 8 months will be the square of its amount for 4 months; its amount for 9 months, the cube of its amount for 3 months; its amount for a year and a quarter will be the product of its amounts for a year, and for 3 months; its amount for 6 years and a half will be the product of its amounts for 6 years, and for half a year; and so in similar cases. All this will appear obvious from a due consideration of the nature of compound interest.

If the interest were payable *half-yearly*, however, at a given rate per cent. *per annum*, as in the third example, the amount at the end of a year would be more than if it were payable yearly. Thus, at 4 per cent. per annum, payable half-yearly, the amount at the end of half a year would be 1·02; at the end of a year, or 2 half-years, $1{\cdot}02^2$, or 1·0404; at the end of a year and a half, or 3 half-years, $1{\cdot}02^3$; at the end of 2 years, or 4 half-years, $1{\cdot}02^4$, &c.

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* If r be put = the amount of £1 for a year, p = the principal, t = the time, and a = the amount; the second rule, expressed algebraically, will become

$$a=pr^t \ldots\ldots (1);$$

or, by taking the logarithms of both members,

$$\log a = \log p + t \log r \ldots\ldots\ldots (2).$$

From equation (1) we have $p=\frac{a}{r^t}$, which is the algebraic expression for Rule III.; also, from equation (2), we get $\log p = \log a - t \log r$, which is a logarithmic formula for the same purpose.

From equation (1) we have also $r=\left(\frac{a}{p}\right)^{\frac{1}{t}}$; and from equation (2), $\log r = \frac{\log a - \log p}{t}$. Each of these serves for the resolution of the problem in which it is required *to find the rate at which a given principal will amount to a given sum in a given time.* The first of them, expressed in words, gives this rule: *Divide the amount by the principal, and that root of the quotient which is denoted by the number of years, will be the amount of £1 for one year; whence the rate will be known.* Thus, if it be required to find at what rate £1000 would amount to £1360 - 17 - 3 in 7 years, let the latter, with its shillings and pence reduced to a decimal, be divided by the former: the quotient is 1·3608625, the *seventh* root of which is 1·045 nearly. Hence, the rate must be $4\frac{1}{2}$ per cent. per annum. The difficulty of the extraction of high roots, renders this mode of calculation inferior almost beyond comparison to that by the logarithmic formula above exhibited, unless, which is rarely the case, a degree of accuracy may be required which cannot readily be obtained by logarithmic tables. In general, indeed, the great facility afforded by logarithms in calculation, will be as much felt in compound interest as in almost any other part of mathematics.

We have also from equation (2), $t=\frac{\log a - \log p}{\log r}$, a formula which will serve *to find the time in which a given principal, at an assigned rate, will amount to a given sum.* This may also be approximated without logarithms by position: for if the given amount be divided by the principal, the quotient will be the amount of £1 for the required time; then this, and the amount of £1 for a year, being known, the time will be found, either exactly, or nearly so, without much difficulty.

Had it been payable quarterly, the amounts at the end of 1, 2, 3, &c., quarters would have been 1·01, $1·01^2$, $1·01^3$, &c. In such cases, *to find the amount of £1 at the end of the proposed time, raise its amount at the end of the first payment, to the power denoted by the number of payments.*

As calculations in compound interest are much facilitated by the use of interest tables, a table, constructed by Rule I., is given at the end of the book, showing the amount of £1 at compound interest, for any number of years not exceeding 50, at the most usual rates. The pupil, after having wrought the preceding exercises by the rules already given, should be taught, instead of finding the amount of £1 by Rule I., to take it, when he can, from the table. This table will also be useful, in many instances, in finding by inspection the time, from the principal, the rate, and the amount; and the rate from the principal, the time, and the amount.

ANNUITIES.

An ANNUITY is a fixed sum of money payable at the ends of equal periods of time, such as years, half-years, or quarters.

Annuities are of two kinds, *certain* and *contingent*.

ANNUITIES CERTAIN are those which commence at a fixed time, and continue for a determinate number of years.

ANNUITIES CONTINGENT are those whose commencement, or continuance, or both, depend on some contingent event, usually the life or death of one or more individuals.

The PRESENT VALUE of an annuity at compound interest, is such a sum as would, if lent at compound interest for the given time, amount to the same sum to which the annuity itself would amount, if *forborne* during the same time; that is, if, instead of being taken up as it becomes due, it were allowed to remain unpaid, and to accumulate at compound interest.*

When an annuity does not come into possession, till a given time has elapsed, or some particular event has taken place, it is said to be an ANNUITY IN REVERSION.

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* An annuity is commonly said to be worth as many *years' purchase* as there are pounds in the present value of an annuity of £1. Thus, in the case of an annuity for 20 years, at 5 per cent. per annum, because the present value of an annuity of £1 is £12·462, &c., the annuity is said to be worth about $12\frac{1}{2}$ years' purchase.

ANNUITIES CERTAIN.

Rule I. *To find the amount of an annuity, payable yearly, the payments of which are forborne for a given time, compound interest being charged on them, as they become due:* (1.) Subtract 1 from the amount of £1 for a year, and from its amount for the given time at compound interest: (2.) Divide the latter remainder by the former, and the quotient will be the amount of an annuity of £1 forborne for the proposed time: (3.) Multiply this amount by the given annuity, and the product will be the amount required.

When the payments are not yearly: Instead of the amount of £1 for a year, use its amount for the interval between the payments; and instead of the number of years, use the number of payments that would have been made during the time they were remitted, and then proceed as before.

Exam. 1. If a person save £120 per annum, and improve it at 5 per cent. per annum, compound interest, how much will he be worth at the end of 20 years?

The amounts of £1 for 1 year and for 20 years, at 5 per cent. per annum, are, as found by Rule I., compound interest, 1·05 and 2·6532977; from each of which, if 1 be subtracted, there remain ·05 and 1·6532977. Let the latter of these be divided by the former, and the quotient 33·065954 is the amount of an annuity of £1 for 20 years; then let this be multiplied by 120, and the product, £3967·91448, or £3967 - 18 - 3½, is the amount required. In this case, the gain by interest is £1567 - 18 - 3½, since the person's savings without interest would have been £120 × 20, or £2400.

Exam. 2. Let everything be as in the last example, except that the annuity is payable *half-yearly*, instead of *yearly*.

Here, since the payments are half-yearly, there would have been 40 payments; and the amount of £1 at the end of half a year, in these circumstances, is £1·025, the 40th power of which is 2·6850723, the amount of £1 at compound interest at the end of 20 years. Then, 1·6850723 ÷ ·025 = 67·402892, is the amount, at the end of 20 years, of an annuity of £1 payable at the end of each period of 6 months. Multiply this by £60, the sum payable each half-year, and the product, £4044·17352, or £4044 - 3 - 5½, is the amount required, which is £76 - 5 - 2 more than the answer of the last question. It is evident, that the more frequent the payments are, the greater is the amount; for the several gains by interest are thus put sooner to gain more interest. (See Exercise 10.)

The theory of the preceding rule, as well as almost every other inquiry in compound interest and annuities, is much more easily and

satisfactorily explained by an algebraic investigation, than it can be by common arithmetic. For the use of those, however, who are unacquainted with algebra, the following illustration of a particular case is annexed. Let it be required to find the amount of an annuity of £1 for 8 years at 5 per cent. per annum, compound interest. At the end of the time the eighth payment would be simply £1; the value of the seventh would be £1·05, as it would remain at interest for 1 year; that of the sixth $£1·05^2$, as it would remain at interest 2 years; that of the fifth $£1·05^3$; of the fourth $£1·05^4$; of the third $£1·05^5$; of the second $£1·05^6$, and of the first $£1·05^7$. Hence, the entire amount to be received at the end of the time would be the sum of the series of continual proportionals, 1, 1·05, $1·05^2$, $1·05^3$, $1·05^4$, $1·05^5$, $1·05^6$, $1·05^7$. But by Rule II., page 271, the sum of this series is $(1·05^8 - 1) \div ·05$, which agrees with the rule here given for finding the amount of an annuity of £1. The rest is obvious.

It may serve to illustrate the nature of annuities, to show another method of resolving the first exercise, which method might also be employed in solving all questions of a similar kind. As £5 : £100 :: £120 : £2400, the principal which would gain £120 per annum. Then, at compound interest, the amount of £2400 for 20 years is £6367 - 18 - $3\frac{1}{2}$; from which £2400 being subtracted, we have remaining £3967 - 18 - $3\frac{1}{2}$, for the interest, or improvement, of this imaginary principal, which is also the amount of the annuity, the same as was found before.

When the pupil has learned to perform the exercises on this rule and the next, he may be taught to use Tables II. and III., at the end of the book, as often as they are applicable. By this means, the labour will often be greatly abridged, in the same manner as operations in compound interest are often shortened by the use of Table I.

Exercises. Find the amounts, at compound interest, of the following annuities, forborne during the given times, and at the given rates per cent. per annum; the first seven being payable yearly:—

Exercises.	£	*s.*	*d.*				*Answers.* £	*s.*	*d.*
1. Annuity	100	0	0	forborne 10 years, at	4	....	1200	12	$2\frac{1}{2}$
2. ———	13	15	9	——— 14 ———	6	....	289	14	$10\frac{3}{4}$
3. ———	56	17	6	——— 9 ———	6	....	653	11	$4\frac{1}{2}$
4. ———	11	7	6	——— 12 ———	5	....	181	1	$1\frac{3}{4}$
5. ———	34	2	6	——— 8 ———	$5\frac{1}{2}$	....	331	14	$11\frac{3}{4}$
6. ———	14	15	9	——— 15 ———	7	....	371	11	$10\frac{3}{4}$
7. ———	51	2	$8\frac{1}{2}$	——— 14 ———	$4\frac{1}{2}$	....	968	2	$0\frac{1}{4}$

Exer. 8. Suppose a person who has a salary of £75 a year, payable half-yearly, to allow it to remain unpaid 17 years: how much will he be entitled to receive at the end of that time, compound interest being allowed at 6 per cent. per annum? *Answ.* £2164 - 17 - $7\frac{1}{2}$.

9. What will be the amount of a salary of £11 - 7 - 6 a year, payable at periods of 2 years each, and forborne for 12 years, at 5 per cent. per annum, compound interest? (See Exer. 4.) *Answ.* £175 - 10 - $7\frac{1}{4}$.

10. Suppose a salary of £120 per annum, payable quarterly, to be forborne at compound interest for 20 years: to what sum will it amount at 5 per cent. per annum? *Answ.* £4083 - 11 - $3\frac{1}{2}$.

RULE II. *To find the present value of an annuity at compound interest:* (1.) Find by the last rule the amount of an annuity of £1 forborne for the given time, and at the given rate: (2.) Divide this by the amount of £1 at compound interest for the given time, and the quotient will be the present value of an annuity of £1 for that time: (3.) Multiply this by the annuity to find the present value.*

In case of an annuity to continue for ever, or, as it is called, a *perpetuity*, subtract 1 from the amount of £1 for a year, or for the interval between the payments, and divide 1 by the remainder: the quotient is the present value of a perpetuity of £1; which multiply by the given perpetuity.

Or, as the given rate is to £100, so is the perpetuity to its present value.

Exam. 3. Required the present value of a house held on a lease of which 22 years are unexpired, and yielding a profit rent of £45 - 10 per annum, payable yearly, compound interest being allowed at 6 per cent. per annum.

Here, the amount of £1 for 22 years, at compound interest, is 3·603537. Then dividing 2·603537 by ·06, we get 43·3923; the quotient of which, by 3·603537, is 12·041583, the present value of an annuity of £1 for 22 years, at 6 per cent. per annum. Let this be multiplied by £45 - 10, and the product is £547·892026, or £547 - 17 - 10, the required value.

Exam. 4. Let everything be the same as in the last example, except that the rent is payable *half-yearly*, instead of *yearly*.

In this case the amount of £1, at the end of 6 months, is 1·029563, the square root of 1·06; the 44th power of which (44 being the

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* Some may perhaps prefer the following very simple rule for this problem: *As the given rate is to £100, so is the annuity to the imaginary principal which would annually produce the annuity. Then, from this principal subtract its present worth, found by Rule* III. *for compound interest: the remainder will be the present worth of the annuity.*

It may be remarked, that when the payments are not yearly, different writers have viewed the subject in different lights, and given different rules for computing the present value. The method employed in Example 4 is perhaps preferable; but the reader who is well acquainted with the principles of compound interest, will find it easy to form other rules founded on different supposition.

number of payments), or its equal, the 22nd power of 1·06, is 3·603537. From this take 1, and divide the remainder, 2·603537, by ·029563, and the result by 3·603537: the quotient, 24·43916, is the present value of £1 of each payment; which being multiplied by £22 - 15, the half-yearly payment, the product is £555·99089, or £555 - 9 - 9¾, the amount required.

Exam. 5. Required the value of a perpetuity of £80 a year, at 6 per cent. per annum.

As £6 : £100 :: £80 : £1333 - 6 - 8; or £80 ÷ ·06 = £1333⅓, the value required.

Exam. 6. Suppose the same perpetuity as in the last question, payable half-yearly: what is its present value?

At 6 per cent. the amount of £1 at the end of half a year, is £1·029563; and in this question each payment is £40: therefore the value of the perpetuity is £40 ÷ ·029563 = £1353·0426, or £1353 - 0 - 10¼, exceeding that found in the last question, by £19 - 14 - 2¼, in consequence of the frequency of the payments.

The *reason* of the first part of this rule is evident from the definition of the present value of an annuity (page 280), and from Rule I. in this article, and Rule III. in compound interest. It might also be easily shown from Rule III. of compound interest, that, at any particular rate, as 5 per cent., the present value of an annuity of £1 is the sum of the decreasing series of continual proportionals, whose terms are the present worths of £1 at compound interest for 1 year, 2 years, 3 years, and so on; the first term being 1 ÷ 1·05, the ratio the reciprocal of 1·05, and the number of terms equal to the number of years. Now, the sum of the *infinite* series of present worths, found by the rule in page 272, agrees with the rule here given for a perpetuity.

Exercises. Find the present values of the following annuities, for the given times, and at the given rates per cent. per annum, compound interest:—

	Exercises. £	*s.*	*d.*						*Answers.* £	*s.*	*d.*
11. Annuity	84	7	9	to continue	9	years, at	5	......	599	16	2¾
12. ———	46	12	9	———	12	———	6	......	391	0	0¼
13. ———	75	0	0	———	6	———	2½	......	413	2	2¼
14. ———	58	10	0	———	5	———	3½	......	264	2	7¼
15. ———	113	15	0	———	10	———	4	......	922	12	3½
16. ———	95	5	6	———	13	———	10	......	676	15	5¼

Exer. 17. How much must a person pay to have a salary of £224 a year for 19 years, being allowed compound interest at 5 per cent. per annum? *Answ.* £2707 - 2 - 2¾.

18. Suppose a widow to be entitled to £40 a year, payable half-yearly, from a fund, for 8 years: what is it worth at present, at 6 per cent. per annum, compound interest? *Answ.* £252 - 1 - 3¼.

19. If a farmer have a lease of 65 acres of land for 36 years, at £1 - 12 per acre, what fine must he pay to reduce the rent to 15*s.* per acre, compound interest being allowed him at 6 per cent. per annum? *Answ.* £807 - 16 - $2\frac{1}{4}$.

20. Required the present values of a perpetuity of £1, and of another of £68 - 5, payable annually, at $4\frac{3}{4}$ per cent. per annum. *Answ.* £21 - 1 - $0\frac{12}{19}$, and £1436 - 16 - $10\frac{2}{19}$.

21. Find the present value of a perpetuity of £96 - 7 - 6, and of another of £1, payable yearly, at $3\frac{1}{2}$ per cent. per annum, compound interest. *Answ.* £2753 - 11 - $5\frac{1}{7}$, and £28 - 11 - $5\frac{1}{7}$.

22. Required the present values, at 5 per cent. per annum, of a perpetuity of £1, payable yearly, payable half-yearly, and payable quarterly. *Answ.* £20; £20 - 4 - $11\frac{1}{4}$; and £20 - 7 - 5.

Rule III. *To find the present value of an annuity in reversion:* (1.) Find, by Rule II., the present value of the annuity from the present time till the end of the period of its continuance: (2.) Find also its value for the time before it comes into possession: (3.) The difference of these two results will be the present value required.

The reason of this rule is so obvious as to require no explanation. The following rule, for the same purpose, is also founded on obvious principles, and may perhaps be preferred by some.

Rule IV. (1.) Find, by Rule II., the present value of the annuity during the time it is to be possessed: (2.) Then, the present value of this result, found by Rule III. of compound interest, will be the present value of the reversion.

The next rule, which is, *in principle*, the same as the last, will perhaps be preferable for the purposes of computation to either of the preceding.

Rule V. Subtract 1 from the amount of £1 at compound interest during the period in which the annuity is to be possessed: divide the remainder by the amount of £1, at compound interest, during the period from the present time till the termination of the annuity; and the quotient by the amount of £1 for a year, diminished by 1; the result will be the present value of a reversion of £1.

Exam. 7. A father leaves to his eldest child, for 8 years, a profit rent of £280 per annum, payable yearly, and the reversion of it, for the 12 years succeeding, to his second child. What is the present value of the legacy of the second, at 4 per cent. per annum, compound interest?

By Rule III. Here, by Rule II., the value of an annuity of £1 for 20 years, at 4 per cent., is 13·590325, and for 8 years, 6·732745; the difference of which is 6·85758, the present value of a reversion of £1 in the proposed circumstances. Multiplying this by £280, we get £1920·1224, or £1920 - 2 - $5\frac{1}{4}$, the value required.

By Rule IV. The present value of an annuity of £1 for 12 years, is found by Rule II. to be 9·385073; and, by Rule III. of compound interest, the present value of this for 8 years is found to be 6·857579, as before.

By Rule V. The amount of £1 for 12 years, diminished by 1, is ·601032; and the amount of £1 for 20 years is 2·191123. Divide the former by the latter, and the result by ·04, and there will finally result 6·857579, the same present value as before.

Exercises. Required the present values of the following annuities in reversion, at the given rates per cent. per annum :—

	Exercises. £	*s.*	*d.*				*Answers.* £	*s.*	*d.*
23.	Ann. 135	10	9,	after 6 yrs.	& for 8 yrs.	at $5\frac{1}{2}$ ...	622	13	6
24.	—— 79	12	6,	—— 4	—— 6	—— 5 ...	332	9	$11\frac{1}{4}$
25.	—— 58	9	10,	—— 3	—— 7	—— $4\frac{1}{2}$ ...	302	0	$8\frac{3}{4}$
26.	—— 54	12	3,	—— 4	—— 8	—— 7 ...	248	15	$8\frac{1}{2}$
27.	A perpetuity of £84	7	6,	after 8		—— 7 ...	701	10	7
28.	—— 136	17	9,	—— 8		—— 5 ...	1853	0	$4\frac{1}{4}$

Exer. 29. What fine must be paid, to change into a perpetuity, a lease for 16 years, which yields a profit rent of £71 - 13 - 3 per annum, payable yearly, compound interest being allowed at $4\frac{1}{4}$ per cent. per annum? *Answ.* £866 - 6 - $8\frac{1}{2}$.

30. What fine must be paid to add 25 years to a lease which brings a profit rent of £112 - 10, and of which 14 years are unexpired, compound interest being allowed at 5 per cent. per annum? *Answ.* £800 - 16 - $4\frac{3}{4}$.

31. What is the present value of the reversion of a perpetuity of £60 per annum, payable yearly, but not to come into possession till the expiration of 100 years, compound interest being allowed at 6 per cent. per annum? *Answ.* £2 - 18 - $11\frac{1}{4}$.

This question may tend to correct a mistake that pretty generally prevails, respecting the comparative values of long leases and perpetuities, the latter being supposed to exceed the former in value, in a far greater degree than they really do. In the case on which this exercise is founded, the difference of the present values is no more than £2 - 18 - $11\frac{1}{4}$.

ANNUITIES CONTINGENT, OR LIFE ANNUITIES.

Life annuities are those whose commencement or termination, or both, depend on the extinction of one or more lives.

When life annuities are in possession, they are often called

simply ANNUITIES ON LIVES; but when they are in reversion, they are generally called REVERSIONARY ANNUITIES.

The VALUE OF A LIFE is the present value of an annuity of £1, to continue during that life.

The calculation of life annuities depends on the joint application of the rules of compound interest, and of the doctrine of chances, to tables deduced from observations on the duration of human life. In what follows on this subject, a selection of the rules most generally useful will be given. For the theory of these rules, which is of a nature too complicated to be given in a work like the present, the reader who wishes to become thoroughly acquainted with this interesting subject may have recourse to the writings of Simpson, De Moivre, Price, Morgan, Baily, Milne, David Jones, &c., and more particularly to the Journal of the Institute of Actuaries, where the science in all its branches will be found treated of at great length by many able writers.

Tables constructed for the purposes of determining the duration of human life and the values of life annuities are called MORTALITY TABLES, or sometimes LIFE TABLES. The duration of life being different in different countries, and in different classes of the population of the same country, mortality tables have been formed from statistical records kept at various places, as in London, Breslau, Northampton, and Carlisle. National tables have also been constructed, such as the English Life Tables, from the census returns and death registers of whole countries. Life insurance companies, too, have frequently made use of the facts derived from their policy books, in order to prepare mortality tables. Tables IV. and V. at the end of the volume, which are employed in what follows, are founded on the experience of twenty of the leading British life offices, which was collected by the Institute of Actuaries, and published in 1869.

RULE I. *To find the present value of an annuity, to continue during the life of a person whose age is given:* Take from Table IV. the value of £1 for the given age and rate, and multiply it by the given annuity.

Exam. 1. What should be given, at 5 per cent. per annum, for a farm worth £36 a year, held on a lease for one life aged 58 years?

Here, by the table, the value of an annuity of £1 on the life of a person aged 58 years, is, at 5 per cent., 9·330, the product of which by 36 is £335·880, or £335 - 17 - $7\frac{1}{4}$, the value required.

Exer. 1. If a person, aged 38 years, have a salary of £138 - 10 a year for life, what is its present value at 5 per cent. per annum? *Answ.* £1911 - 19 - $10\frac{1}{4}$.

2. What should a person, aged 32 years, pay, at 4 per cent. per annum, to have for life a yearly salary of £180? *Answ.* £3019 - 6 - $4\frac{3}{4}$.

3. If a farm of 28 acres be held at £1 - 14 - 6 per acre, on a life aged 41 years, what fine must be paid, at 5 per cent. per annum, to reduce the rent to 10 shillings per acre? *Answ.* £455 - 14 - $10\frac{1}{2}$.

RULE II. *To find the present value of an annuity which is to continue during* the joint lives *of two persons, and to cease when either of them dies:* In Table V., find the age of the younger, or of either, if they be equal, in the first column; and in the same division of the table, in the second column, find the age of the other; opposite to the latter is the value of an annuity of £1, which multiply by the given annuity.

Exam. 2. What is the value, at 4 per cent., of an annuity of £90 per annum, to continue during the joint lives of two persons, whose ages are 15 and 50 years respectively?

Here, by Table V., the value of an annuity of £1 is £11·805, the product of which by £90 is £1062·450, or £1062 - 9 - 0, the value required.

Exam. 3. Required the present value, at 4 per cent., of an annuity of £120 per annum, which is to cease when either of two persons, aged 14 and 57 years respectively, shall die.

Neither of these ages being in the table, recourse must be had to the method of proportional parts. (See page 251.) Thus, the table gives, for 10 and 55, 10·591, and for 15 and 55, 10·492. The difference of these is ·099, four fifths of which being subtracted from 10·591, the remainder, 10·512, is the value of £1 for the ages, 14 and 55. In like manner, the table give for 10 and 60, 9·135, and for 15 and 60, 9·063; four fifths of the difference of which being taken from 9·135, the remainder, 9·077, is the value of £1 for the ages, 14 and 60. Hence, we have, for 14 and 55, 10·512, and for 14 and 60, 9·077; two fifths of the difference of which being subtracted from 10·512, the remainder, 9·938, is very nearly the present value of £1 for the ages 14 and 57. and £9·938 being multiplied by 120, the product, £1192·56, or £1192 - 11 - 2½, is the value required.

Exer. 4. Required the present value, at 4 per cent., of an annuity of £39 - 10, on the joint continuance of two lives of 25 and 73 years. *Answ.* £206 - 0 - 7¾.

RULE III. *To find the present value of an annuity to continue during the* longer *of two lives:* From the sum of the values of the single lives, found in Table IV., subtract the value of the joint lives, found by Rule II.; the remainder is the present value of an annuity of £1 on the longer of the two lives.

Exam. 4. For how much should a house be sold which yields a profit rent of £41 - 5 per annum, and is held by a lease on the longer of two lives aged 20 and 35 years, compound interest being allowed at 4 per cent. per annum?

By Table IV., the values of the lives are 18·644 and 16·197, the sum of which is 34·841. Also, by Table V., the value of them jointly is 14·504; which taken from the preceding sum, leaves 20·337, the value of an annuity of £1 on the life of the longer liver; £20·337, multiplied by 41·25, gives £838·901, or £838 - 18 - 0¼, the present value required.

Exer 5. What should a man, aged 44, pay at 4 per cent., to

secure during his own life, and that of his wife,* aged 39, an annuity of £200 a year? *Answ.* £3528 - 0 - 0.

RULE IV. *To find the value of an annuity during the joint continuance of three lives; or which is to terminate on the extinction of any one of them:* Find, by Rule II., the value of the two eldest jointly; and, by Table IV., find what single life would have this same value. Then find, by Rule II., the value of the joint continuance of the single life thus found, and of the youngest, and this will be the value of the three proposed lives jointly.

Exam. 5. Required the present value, at 4 per cent., of an annuity of £140 on the joint continuance of three lives, aged 15, 30, and 35 years.

Here, at 4 per cent., the value of the joint continuance of two lives of 30 and 35, is 13·908; which, in Table IV., is found to be the value of a single life of 45 years nearly. Then, by Rule II., the value of the joint continuance of two lives of 15 and 45, is 12·968, the value of an annuity of £1 on the joint continuance of the three given lives. The product of £12·968 by 140 is £1815·52, or £1815 - 10 - 4¾, the value of the given annuity.

RULE V. *To find the value of an annuity on the* longest *of three lives; or which is to continue, till they are all extinct:* From the sum of all the values of the single lives, found in Table IV., and of the value of the three jointly, found by Rule IV., subtract the sum of the joint values of the lives combined two and two, by Rule II.; and multiply the remainder by the given annuity.

Exam. 6. Required the present value, at 4 per cent., of a house and farm which yield a profit rent of £32 - 10 per annum, held by a lease of three lives, aged 35, 30, and 15 years.

Here, by the last example, the value of the three lives jointly is 12·968, and their values singly are 16·197, 17·131, and 19·417. Also, by Rule II., the value of the first and second jointly is 13·908; of the first and third, 14·819; and of the second and third, 15·516. From the sum of the first four of these numbers take the sum of the others, and the remainder, 21·470, is the value of an annuity of £1 on the life of the longest liver of the three. £21·470 multiplied by 32·5, gives, for the value required, £697·775, or £697 - 15 - 6.

Exer. 6. How much should a man pay, at 4 per cent. per annum, to purchase an annuity of £320 a year, to continue till himself, aged 55, his wife, aged 50, and his son, aged 20, shall all be extinct? *Answ.* £6313 - 12 - 0.

RULE VI. *To find the value of the reversion of an annuity after the death of the present possessor:* From the present value of an annuity of £1 for the entire continuance of the annuity, subtract the value of an annuity of £1 during the life of the possessor: the re-

* The duration of the lives of females is found to be somewhat greater than that of males. This, however, has not been taken into consideration in the solution of Exercise 5.

mainder will be the value of a reversion of £1 in the given circumstances; which multiply by the given annuity.

If there be two or three lives, the same rule will serve, their values being used instead of the value of the single life.

Exam. 7. If a person, aged 47 years, possess a perpetuity of £500 per annum, what is its present value, at 4 per cent., to his son, who is to possess it at his death?

Here, the value of £1 in perpetuity is £25; and, by Table IV., the value of an annuity of £1 on a life of 47 years is 13·366. Then, £25 – £13·366 = £11·634; the product of which by 500 is £5817, the value required.

Exam. 8. A man leaves an annuity of £165 per annum, of which 30 years are yet to come, to his nephew, aged 38 years, during life, or till its termination; and the reversion of it to a charitable institution, in case the nephew die before the termination of the annuity. How much is the present value of the reversion at 4 per cent.?

In this case, the present value of an annuity of £1 for 30 years, is, by Rule II. of annuities certain, 17·292; and, by Table IV., the present value of £1 on a life of 38 years, is 15·575. Then, £17·292 – £15·575 = £1·717; and £1·717 × 165 = £283·305, or £283 - 6 - $1\frac{1}{4}$, the value required.

Exer. 7. What is the present value, at 4 per cent., of the reversion of a perpetuity of £400 per annum, not to come into possession till the death of the present incumbent, aged 70, and of his intended successor, aged 30? *Answ.* £3027 - 4 - 0.

Rule VII. *To find the present value of a proposed sum payable at the decease of a person whose age is given:* From the value of £1 in perpetuity, take the value of an annuity of £1 on the life; divide the remainder by the value of the perpetuity increased by 1; and multiply the result by the given sum.

If the present value thus found be divided by the value of the life, the quotient will be the annuity payable yearly.

By using the value of two or three lives, found by the preceding rules, we may apply the same rule.

Exam. 9. How much must a person, aged 32 years, pay, at 4 per cent., to an insurance office, to secure £1000 to his family at his death?

At 4 per cent., the perpetuity of £1 is £25; and, by Table IV., the value of a life of 32 years is 16·774. The difference of these is 8·226; and this being divided by 26, the quotient is ·3164, the present value of £1 payable at the decease of the person, Then, £·3164 × 1000 = £316 - 8 - 0, the value required: and £316·4 ÷ 16·774 = £18·863 = £18 - 17 - 3, the annuity. Hence it appears, that at 4 per cent. per annum, if a person, aged 32 years, either make one payment of £316 - 8 - 0, or pay each year during life £18 - 17 - 3, his heirs will be entitled to receive £1000 at his death.

Exer. 8. At 3 per cent. per annum, how much yearly must a person, aged 66 years, pay during life, to entitle his heirs at his death to receive £1500? *Answ.* £136 - 18 - 0.

9. If a husband and his wife be each aged 45 years; how much per annum must be paid during the longer of the two lives, to entitle the family, after the decease of both, to receive £4000, compound interest being allowed at 3 per cent. per annum? *Answ.* £87 - 12 - 2½.

Exam. 10. How much must a man, aged 39, pay per annum at 3 per cent., during his marriage, or during life, to entitle his wife, aged 32 years, in case she survive him, to an annuity of £50 a year during the remainder of her life?

By Rule II., the value of the two lives jointly is 14·790, and the value of a life of 32 years is, by Table IV., 19·373. The difference of these, 4·583, is the present value of the reversion of an annuity of £1 to be paid to the wife after her husband's death, in case she survive him. The product of £4·583 by 50 is £229·15, or £229 - 3 - 0, the sum to be paid at a single payment. Let this be divided by 14·790, the value of their lives jointly, the quotient, £15·494, or £15 - 9 - 10½, is the annual payment. Hence, if there be paid annually, during the joint continuance of both lives, the sum of £15 - 9 - 10½, or at present, in a single payment, £229 - 3 - 0, the wife, if she survive the husband, will be entitled to an annuity of £50 per annum during life.

The above solutions under Rule VII. proceed on the supposition with regard to the annual payments, that the first of these was not made till the end of the first year. If it be made at present, however, we must divide, in each case, by the value of the life or lives increased by unity.

Exam. 11. If a number of persons form themselves into an association for providing annuities of £40 per annum for their surviving widows, and each person, at the age of 30, contribute £75 to the fund and equal annual contributions during the rest of his life, how much must each of these annual contributions be, if money be improveable at 4 per cent. per annum, and if the wife of each be supposed to be at an average three years younger than himself?

The value of a life of 27 years is 17·641, and the value of two lives of 27 and 30 jointly is 14·693. The difference of these, £2·948, multiplied by 40, gives for the entire sum to be paid at present, at a single payment, £117·92. But, as only £75 is to be paid at present, there will still remain £42·92 to be paid by annual contributions; by dividing which by 14·693, we obtain £2 - 18 - 5 for each of these contributions.

Exam. 12. Suppose that, as in the last question, a man aged 30 years pays £50 into a fund; what annual contribution must he pay during life, to entitle his family to an annuity of £40 per annum, for 8 years after his death, interest being at 4 per cent.?

The present value of an annuity of £40 for 8 years at 4 per cent. is £269·31, by Rule II. of annuities certain. Then, by Rule VII. of this article, the present value of this sum is found to be

£81·509; from which £50 being subtracted, the remainder, £31·509, is the sum still remaining to be paid at present; and this being divided by 17·131, the value of a life of 30, the quotient, £1 - 16 - $9\frac{1}{2}$, is the annual contribution required.

CONTINUED FRACTIONS.*

It frequently happens that fractions, even when reduced to their simplest forms, are expressed in numbers inconveniently large; and hence it is often desirable to approximate their values in smaller numbers. Thus, if the fraction $\frac{1115}{2447}$, which is in its lowest terms, be proposed, we may wish, even for the purpose of forming a more correct idea of its magnitude, to find other fractions, in smaller terms, which will be nearly of the same value. The method that most naturally presents itself for this purpose, is to divide both terms by the smaller of them, as the smaller will by this means become 1, and we shall thus be enabled to compare the other with that number with which the mind can most easily compare it with respect to magnitude. By this means, the fraction becomes $\dfrac{1}{2\frac{217}{1115}}$; the denominator of which being between 2 and 3, we conclude that the value of the given fraction is between $\frac{1}{2}$ and $\frac{1}{3}$: and therefore $\frac{1}{2}$ is a first approximation to its value, being too great. Divide, again, both terms of the fraction in the denominator by 217, and it will become $\dfrac{1}{5\frac{30}{217}}$, which is between $\frac{1}{5}$ and $\frac{1}{6}$. By taking $\frac{1}{5}$ instead of $\frac{217}{1115}$, we shall have for a second approximate value of the proposed fraction, $\dfrac{1}{2\frac{1}{5}}$, or $\frac{5}{11}$, which is too small, as instead of $\frac{217}{1115}$, we used in the denominator $\frac{1}{5}$, which is greater than $\frac{217}{1115}$. To continue the approximation still farther, we divide both terms of the fraction $\frac{30}{217}$ by 30: the result is $\dfrac{1}{7\frac{7}{30}}$, which is less than $\frac{1}{7}$ and greater than $\frac{1}{8}$. Hence, if instead of using $\frac{1}{5}$ in the preceding fraction, we use $\dfrac{1}{5\frac{1}{7}}$ or its equal $\frac{7}{36}$, we get for the next approximation, $\dfrac{1}{2\frac{7}{36}}$ or $\frac{36}{79}$, which is too great, because $\dfrac{1}{5\frac{1}{7}}$, or its equal $\frac{7}{36}$, is too small, in consequence

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* Full information respecting the nature and applications of continued fractions is to be found in Lacroix, *Complément des Elémens d'Algèbre*; Legendre, *Essai sur la Théorie des Nombres*; Barlow's *Theory of Numbers*; Lagrange's *Additions to Euler's Algebra*, or in modern English treatises on algebra generally. What is here given is one of the most interesting applications, and one which will be useful to the more advanced arithmetician.

of $\frac{1}{7}$ being too great; and therefore $\frac{1}{2\frac{7}{36}}$ is too great, since the denominator is too small. By continuing the process in a similar manner, we find $\frac{149}{327}$, $\frac{483}{1060}$, and $\frac{1115}{2447}$, for the succeeding fractions, the last of which is the given fraction, and the first an approximate fraction smaller, and the second another greater, than the given fraction. If the given fraction be finite, the last of the converging fractions will always be equal to it, as in this example. These successive fractions have the remarkable property that *each of them approaches more nearly than the one which precedes it, to the value of the given fraction.* Thus, $\frac{1115}{2447}$ is nearly equal to $\frac{1}{2}$; more nearly equal to $\frac{5}{11}$; more nearly to $\frac{36}{79}$; still more nearly to $\frac{149}{327}$; and, finally, more nearly still to $\frac{483}{1060}$. That there is this continual approach to the true value of the fraction, will appear evident, if it be considered, that in each of the results in the preceding operations, a correction is made on the result which goes before it.

The several converging fractions above obtained, if the last or supplementary simple fraction be rejected from each, except when its numerator, like that of the rest, is 1, may be written at full length, as follows:—

$$\frac{1}{2}; \quad \cfrac{1}{2+\cfrac{1}{5}}; \quad \cfrac{1}{2+\cfrac{1}{5+\cfrac{1}{7}}}; \quad \cfrac{1}{2+\cfrac{1}{5+\cfrac{1}{7+\cfrac{1}{4}}}};$$

$$\cfrac{1}{2+\cfrac{1}{5+\cfrac{1}{7+\cfrac{1}{4+\cfrac{1}{3}}}}}; \quad \cfrac{1}{2+\cfrac{1}{5+\cfrac{1}{7+\cfrac{1}{4+\cfrac{1}{3+\cfrac{1}{2}}}}}}.$$

In this form they are called *continued fractions.* It appears, therefore, that a CONTINUED FRACTION *is that whose denominator is a whole number with a fraction annexed; which fraction has also for its denominator a whole number with a fraction, the denominator of which latter fraction is a whole number with a fraction; and so on, however far the fraction may be continued; and each numerator is* 1.*

It will be seen by a review of the preceding processes, that the denominators of the continued fractions are the quotients which would be found in using the rule given in page 132, for finding the greatest common measure. Hence, we have the following rule:—

RULE I. *To convert a given simple fraction into a continued one:* Divide the greater term by the less, the less by the remainder, &c., as in finding the greatest common measure: the quotients will be the denominators of the several fractions in the continued fraction, and the numerator of each will be 1.

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* It is not essential to the nature of continued fractions that each numerator be unity; but fractions of this kind only are used as instruments of calculation.

The successive fractions which approach continually to the value of a given fraction expressed in large numbers, may be found by reducing it to a continued fraction, and operating in the way already employed in approximating to the value of $\frac{1115}{2447}$: the following method, however, will be found much preferable.

RULE II. *A fraction expressed by a great number of figures being given, to find all the fractions in less terms, which approach so near the truth, that it is impossible to approach nearer without employing greater terms:* Find the several quotients, by the preceding rule. Then, write the several quotients in a line; and if the given fraction be greater than 1, take the first quotient as the numerator, and 1 as the denominator, of the first fraction, which set below the second quotient; but if the given fraction be less than 1, make the first quotient the denominator, and 1 the numerator of the first fraction. Then, for the second fraction, multiply both the terms of the first by the quotient which stands above it, and add 1 to the product of that term which was the first quotient: the result is the second fraction, which is to be set below the third quotient. To find the succeeding fractions, multiply the terms of each fraction, when found, by the quotient which stands above it, and to the products add separately the terms of the preceding fraction.

Exam. 1. The height of Slieve Donard, in the county of Down, is 2654 feet, and that of Ben Lomond 3262 feet: it is required to find a series of converging fractions, expressing as nearly as possible the ratio of the heights of these mountains.

Here, the fraction expressing the height of Slieve Donard in relation to that of Ben Lomond, is $\frac{2654}{3262}$, or, in its lowest terms, $\frac{1327}{1631}$, and the quotients, found in the manner shown in the margin, are 1, 4, 2, 1, 2, 1, 4, 1, 4: and the successive converging fractions will be found thus:—

```
1327)1631(1
     1327
      304)1327(4
          1216
           111)304(2,
               &c.
```

$$\begin{array}{ccccccccc} 1 & 4 & 2 & 1 & 2 & 1 & 4 & 1 & 4 \\ \frac{1}{1}, & \frac{4}{5}, & \frac{9}{11}, & \frac{13}{16}, & \frac{35}{43}, & \frac{48}{59}, & \frac{227}{279}, & \frac{275}{338}, & \frac{1327}{1631}. \end{array}$$

Here, the quotients, or denominators, being arranged in succession, we take, for the denominator of the first fraction, 1, the first quotient, and we make its numerator also 1. Then, by multiplying both terms by 4, the quotient which stands above them, and adding 1 to the product resulting from the denominator, because it was the first quotient, we obtain $\frac{4}{5}$ for the second fraction. We then multiply 4 by 2, the figure above it, and add to the product the preceding numerator; we also multiply 5 by 2, and add to the product the preceding denominator; and we thus obtain $\frac{9}{11}$ for the third fraction. We next multiply the terms of this fraction by 1, and adding to the results 4 and 5 severally, we find the next fraction to be $\frac{13}{16}$. In this way, rejecting the first fraction, $\frac{1}{1}$, which is evidently far from the truth, we find that the height of Slieve Donard is nearly $\frac{4}{5}$ of that of Ben Lomond; more nearly $\frac{9}{11}$ of

it; still more nearly, $\frac{13}{18}$; more nearly still, $\frac{35}{43}$, &c.; $\frac{4}{5}$ being too small, $\frac{9}{11}$ too great, $\frac{13}{18}$ too small, $\frac{35}{43}$ too great, &c.*

Exam. 2. The circumference of the circle whose diameter is 1, is found to be greater than 3·1415926, but less than 3·1415927: required the series of fractions converging to the ratio of the diameter and the circumference.

In questions like this, in which one term of the fraction or ratio is not precisely given, but is contained between given limits, as when one of the terms is an infinite decimal, it is proper to work for the quotients by both limits, and to use those only which result from both. Thus, in the proposed exercise, by dividing 3·1415926 by 1, this divisor by the remainder, &c.; and by proceeding in like manner with 3·1415927, we find in both cases the quotients, 3, 7, 15, 1, after which the quotients would be different, and are therefore not to be used. Hence, the converging fractions are found as in the margin.

$$\begin{array}{cccc} 7 & 7 & 15 & 1 \\ \frac{3}{1}, & \frac{22}{7}, & \frac{333}{106}, & \frac{355}{113}. \end{array}$$

The diameter of a circle, therefore, is to its circumference nearly as 1 to 3; more nearly, as 7 to 22; more nearly again, as 106 to 333; and still more nearly, as 113 to 355. The degree of approximation of each of these to the given ratio will be discovered by dividing the second term of each by the first. In this way the last gives 3·14159292, which exceeds the truth by less than a ten-millionth part of the circumference. Had the circumference been taken to more places (see note, page 189), the succeeding fractions would have been found to be $\frac{103993}{33102}$, $\frac{104348}{33215}$, &c.

Exam. 3. Let it be required to approximate the ratio of 1·41421† to 1.

Here the quotients are found to be 1, 2, 2, 2, 2, &c.; and consequently the continued fraction is what is in the margin. The converging fractions, also, are thus found:—

$$1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\text{\&c.}}}}}$$

$$\begin{array}{cccccc} 1 & 2 & 2 & 2 & 2 & 3 \\ & \frac{1}{1}, & \frac{3}{2}, & \frac{7}{5}, & \frac{17}{12}, & \frac{41}{29}, \; \frac{99}{70}, \text{ \&c.} \end{array}$$

* Thus, $\frac{4}{5}$ of $3262=2609\frac{3}{5}$; $\frac{9}{11}$ of $3262=2668\frac{10}{11}$; $\frac{13}{18}$ of $3262=2650\frac{3}{8}$; $\frac{35}{43}$ of $3262=2655\frac{5}{43}$; $\frac{48}{59}$ of $3262=2653\frac{49}{59}$, which are evidently approaching 2654, the true value, and are alternately greater and less.—It may be observed, that in all cases *the error of each fraction is a less part of the integer than unity divided by the product of its own and the succeeding denominator; but a greater part than unity divided by the product of its own denominator, and the sum of that and the succeeding denominator.* Thus, in the preceding example, the error of $\frac{4}{5}$ is less than $\frac{1}{55}$ of 3262, but greater than $\frac{1}{80}$ of the same.

† This is the square root of 2, which therefore is expressed by a unit, with a continued fraction, each of whose denominators is 2. Hence the fraction may be continued without limit, the law of continuation being manifest, which is not the case when the root is expressed decimally. The same, as can be shown by algebra, holds respecting the square root of every number which is not a square. Thus, the root of 11 is expressed by the continued fraction in the margin, the law of continuation of which is manifest.

$$\sqrt{11}=3+\cfrac{1}{3+\cfrac{1}{6+\cfrac{1}{3+\cfrac{1}{6+\text{\&c.}}}}}$$

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ımes would be invented, by some process of formation now un-known, for all the numbers so far as ten, and also for the powers of ten, the names of all other numbers would be obtained by a proper combination of these. Thus, the number eleven might be called *one and ten*; twelve, *two and ten*; thirteen, *three and ten*; twenty, *two tens*; thirty, *three tens*, &c.; and we find this method of denominating numbers strictly followed, except in some of the smaller numbers, such as eleven, twelve, twenty, thirty, &c.; which, from their frequent use, have been more liable to have their names corrupted and altered, but which, when their derivations can be discovered, are always found to be formed on correct, analogical principles, according to the foregoing explanation.

Of the advantages and excellence of this system of notation, we can scarcely be duly sensible. Instructed in its use from the earliest notions we receive in arithmetic; never comparing it, or comparing it but slightly, with other modes of expressing numbers by characters; and finding no deficiency, no need of improvement, nothing to call our thoughts to the subject, we use it without feeling its superiority, and with a very inadequate idea of its power. We do not reflect, that merely by means of the different positions and combinations of no more than ten simple characters, we can correctly and easily express any number, however great. With the Roman, or even with the Greek notation, on the contrary, we cannot express numbers that exceed a certain magnitude: and, even were additional characters or marks formed to supply this defect, we should find that calculations, which are performed with great facility and despatch by the decimal notation, would, by either the Greek or Roman system, be excessively tedious and intricate,* while the performance of many would be almost impracticable.

But, though the decimal system of notation has so far the advantage over these, and all other systems not depending on the same principle, we must not conclude that no other system of equal excellence could be invented. There may be an indefinite number of systems, founded on the same principle, and possessing various degrees of excellence. In the decimal notation, we distribute numbers into classes or parcels of *ten* each; these into higher classes, each containing *ten* of the lower; these into still higher classes, each containing *ten* classes of the second order; and so on, till the numbers are exhausted. But if we proceeded still on the same principle, only making the classes consist of *two*, instead of *ten* each, we should have what is called the *binary* scale of notation, in which only the *two* characters, 1 and 0, are requisite for expressing all numbers: and if the classes were made to consist of *three* each, we should have the *ternary* scale, in which only *three* characters, 1, 2, 0, are employed. In the same manner, it is obvious, we might have a *quaternary, quinary, duodecimal, trigesimal, sexagesimal,*

* If the learner should attempt, for instance, by means of the Roman notation, to square 1728 (M.DCC.XXVIII.), which is effected in a few seconds by the modern process, he would have some idea of the immense superiority of the system above the other.

centesimal, or any other scale, by merely taking 4, 5, 12, 30, 60, 100, or any other assigned number, as the number contained in each class. This number may be called the RADIX, the ROOT, or the BASE, of the system; and it is obvious, that in each system there will be as many distinct characters required as there are units in the radix. Thus, in the decimal scale, *ten* characters are necessary, but in the duodecimal *twelve* would be required, which number would be made up by adding to the characters at present in use two others to denote *ten* and *eleven*. In what follows, D will be used to denote *ten*, and H to denote *eleven*; * and, in the duodecimal scale, twelve will of course be written 10.

From these principles, we have, obviously, the following rule:—*To express a given number in any assigned scale:* Divide the given number by the radix of the scale; divide the result also by the radix, and the result arising from this again by the radix. Continue the division in this manner as long as possible, and to the final quotient annex the several remainders in a retrograde order, placing ciphers where there is no remainder. Thus, 592835 in the decimal scale, will be expressed by 2470DH in the duodecimal scale, as will appear from the annexed operation.

12)592835
12)49402......11, or H
12)4116......10, or D
12)343......0
12)28......7
2......4

Here it is evident, that by dividing the given number by 12, it is distributed into 49402 classes, each containing 12, with the remainder 11. By the second division by 12, these classes are distributed into 4116 classes, each containing 12 times 12, or the second power of 12, with a remainder of 10 of the former classes, each containing 12. By the third operation, the classes last found are distributed into 343 classes, each containing 12 of the latter, which were each the second power of 12; and therefore these are each the third power of 12; and the remainder is 0. In like manner, the next quotient expresses 28 times the fourth power of 12, and the remainder, 7 times the third power of 12; and the final quotient expresses twice the fifth power of 12, with a remainder of 4 times the fourth power of 12. Hence, the given number is analysed into $2 \times 12^5 + 4 \times 12^4 + 7 \times 12^3 + 0 \times 12^2 + 10 \times 12 + 11$, or 2470DH, according to the notation above adopted.

It would be easily found by proceeding in the same manner, that for seven thousand, eight hundred, and fifty-four, the expression in the *binary* scale would be 1111010101110; in the *ternary*, 101202220; in the *quaternary*, 1322232; in the *quinary*, 222404; in the *senary*, 100210; in the *septenary*, 31620; in the *octary*, 17256; in the *nonary*, 11686; in the *undenary*, or *undecimal*, 59D0; in the *duodenary* or *duodecimal*, 4666; in the *vigesimal*, or that

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* These characters, which may serve the intended purpose as well as any others in the few instances in which they will here be employed, may be easily recollected by conceiving the first to be formed by running together 1 and 0, the characters which express 10 in the common notation, and by joining with a line 1 and 1, which express eleven.

whose radix is twenty, (19)(12)(14); in the *trigesimal* (radix thirty), 8(21)(24); in the *quinquagesimal* (radix fifty), 374; in the *sexagesimal* (radix sixty), 2D(54); and in the *centesimal* (radix one hundred), (78)(54): where each pair of the figures enclosed in brackets would be represented by a single character, were there a sufficient number of distinct characters for each scale.

The converse of this problem, or *the reduction of a number to the decimal scale from any other*, will be performed by finding the values of the several digits, and collecting those values into one sum; or, more easily, by multiplying the left-hand digit by the radix, and adding to the product the next digit; then by multiplying this sum by the radix, and adding to the product the next digit, and so on, till all the digits shall have been employed.* Thus, 4503142 in the senary scale, is equivalent to $4 \times 6^6 + 5 \times 6^5 + 0 \times 6^4 + 3 \times 6^3 + 1 \times 6^2 + 4 \times 6 + 2$, or 226214; which result will be obtained more easily by the operation in the margin.

$$\begin{array}{r} 4503142 \\ 6 \\ \hline 29 \\ 6 \\ \hline 174 \\ 6 \\ \hline 1047 \\ 6 \\ \hline 6283 \\ 6 \\ \hline 37702 \\ 6 \\ \hline 226214 \end{array}$$

It does not suit the plan or the limits of the present work, to add much to what has been said on this curious and interesting subject. For farther information, recourse may be had to Barlow's "Theory of Numbers," or to any good modern treatise on algebra.

With regard to different scales, it may be sufficient to observe, that the *binary* is chiefly important from its unfolding some curious properties of numbers; that from its employing no character of higher value than unity, operations would be performed by it, though tediously, yet with great facility, and with little mental labour; but an insuperable obstacle to the general use of this scale, is, that in expressing numbers, the characters 1 and 0, must generally be repeated a great number of times. The same advantages, and the same disadvantages belong, but in a less degree, to the *ternary* scale, and still less to the *quaternary*. The *quinary*, *septenary*, and *undenary* all labour under the disadvantage of having their bases prime numbers, and of thus giving origin to a great number of interminate fractions.† The *octary* and *nonary* scales are not so objec-

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* It is scarcely necessary to remark, that both this rule and the preceding are the same in principle, as the rules for the reduction of quantities of different denominations.

† That is, interminate fractions having for their denominators the radix of the scale, or its powers, such as periodical decimals in the common scale.

The numbers which, when used as denominators in the common notation, do not give origin to interminate decimals, are the powers of 2 and 5, and the products of those powers; such as 2, 4, 8, &c.; 5, 25, &c.; 10, 20, 40, &c.; 50, 100, &c. In the senary and duodenary scales, the corresponding denominators are the powers of 2 and 3, and the products of those powers; such as 2, 4, 8, &c.; 3, 9, 27, &c.; 6, 12, 24, &c.; 18, 36, 72, &c. Now, it is easy to show, that, in any considerable interval, there will be almost half as many more numbers of this kind in either of the latter scales, as there are in the decimal scale, the ratio

tionable on this account, but they have no advantage of a different kind to recommend them. The decimal scale does not give origin to so many interminate fractions as either of the latter, and has besides the advantage of expressing numbers rather more concisely. The *senary* and *duodenary* scales, having each so many integral aliquot parts in proportion to its magnitude, and those of so convenient a kind, give origin to much fewer interminate fractions, than any of the above-mentioned. These two are preferable, therefore, in a considerable degree, to any of the others that have been mentioned. The duodecimal has the advantage of expressing numbers concisely, saving one figure in fourteen or fifteen, as compared with the decimal scale; while a number expressed by four figures in the decimal scale, will ordinarily require five in the senary. This slight want of conciseness, however, in the latter, is perhaps more than counterbalanced by the greater facility with which operations would be performed by its means. The largest character employed in it would be 5; while, in the duodecimal scale, all that are used in the decimal one would be required, and two additional ones for ten and eleven: and hence it is easy to see how much more laborious the operations would be in the latter, and how much greater the chance of committing errors. To introduce either of these scales now, however, when men are accustomed to the decimal scale; when the languages of all civilized nations are suited to it; and when so many valuable works, particularly tables, in which it is adopted, would be rendered comparatively useless, would be unadvisable, and perhaps impracticable: but we must regret that the decimal scale was adopted at a time, when any other might have been introduced with equal facility.

The following exercises are annexed for the use of those who may wish to familiarize themselves with this subject:—

Exer. 1. Reduce 1,000,000 in the decimal scale, to the ternary scale, and also to the nonary. *Answ.* 1,212,210,202,001, and 1,783,661.

2. Reduce 1,000,000 in the quaternary scale, to the denary and binary scales. *Answ.* 4096; and 1,000,000,000,000.

3. Reduce 123,454,321 in the senary scale, to the duodenary scale. *Answ.* 987,3D1.

4. How will 476,897 in the decimal scale, be expressed in the duodecimal scale? *Answ.* 1DH,H95.

5. How will 6666 in the decimal scale, be expressed in the binary and quinary scales? *Answ.* 1,101,000,001,010; and 203,131.

6. Reduce 13579 in the duodecimal scale, to the undecimal scale. *Answ.* 190D3.

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being that of the logarithms of 5 and 3. Thus, under 100 there are thirteen such numbers in the decimal scale, and nineteen in the senary or duodenary; while, under 1000, there are twenty-seven in the one scale, and thirty-nine in each of the other two.

When the radix is a prime number, the only denominators of the kind which we are considering, are the powers of the radix. Hence, in the septenary scale, there will be only three such denominators, 7, 49, and 343, below 1000, and in the undenary only two, 11 and 121.

QUESTIONS, WITH THEIR SOLUTIONS.*

1. What number taken from the square of 48, will leave 16 times 54?

Solution. $48^2 = 2304$, and $54 \times 16 = 864$; then $2304 - 864 = 1440$, the answer.

2. Divide £1000 among A, B, and C, and give A £120 more than C, and C £95 more than B.

Sol. A's share is evidently to be 215 ($= 120 + 95$) more than B's; therefore $1000 - 215 - 95 = 690$; and $690 \div 3 =$ £230 = B's share. Hence, $230 + 95 =$ £325, C's share; and $325 + 120 =$ £445, A's share.

3. A father left to his eldest son $\frac{44}{79}$ of his property, to his second $\frac{35}{79}$ of the remainder, and to his third son what was left. What was the share of each, the shares of the first and second differing by £500?

Sol. $1 - \frac{44}{79} = \frac{35}{79}$, and $\frac{35}{79}$ of $\frac{35}{79} = \frac{1225}{6241}$, the part of the whole property belonging to the second son. Then, $\frac{44}{79} + \frac{1225}{6241} = \frac{4701}{6241}$, and $1 - \frac{4701}{6241} = \frac{1540}{6241}$, the share of the youngest. Also $\frac{44}{79} - \frac{1225}{6241} = \frac{2251}{6241}$, the difference of the shares of the first and second. Then, as $\frac{2251}{6241} : \frac{44}{79}$, or as $2251 : 3476 ::$ £500 : £$772\frac{228}{2251}$, the share of the eldest, so is $2251 : 1225 ::$ £500 : £$272\frac{228}{2251}$, the share of the second: and is $2251 : 1540 ::$ £500 : £$342\frac{158}{2251}$, the share of the third.

4. A gets £4 of a legacy for £3 that B gets, and C £5 for £6 that B gets, and A's share is £5000. What is the whole legacy?

Sol. As £3 : £6 :: £4 : £8, A's part, when B gets £6, and C £5. Then as $8 : 8 + 6 + 5 ::$ £5000 : £11,875, the whole legacy.

5. A person possessed of $\frac{3}{8}$ of a ship, sold $\frac{2}{3}$ of his share for £1260. What is the value of the whole ship at the same rate?

Sol. $\frac{2}{3}$ of $\frac{3}{8} = \frac{1}{4}$; and as $\frac{1}{4} : 1 ::$ £1260 : £5040, the answer.

6. A person being asked the hour of the day, said, that the time past noon was $\frac{4}{5}$ of the time till midnight. What was the hour?

Sol. As $1 + \frac{4}{5} : \frac{4}{5}$, or as $9 : 4 :: 12$ hours : 5 hours, 20 minutes; so that the time was 20 minutes past 5 o'clock in the afternoon.

7. A, B, and C purchase a ship; A pays $\frac{2}{9}$, B $\frac{2}{7}$, and C £2000, of the cost. What are the sums paid by A and B?

Sol. $\frac{2}{9} + \frac{2}{7} = \frac{32}{63}$, and $1 - \frac{32}{63} = \frac{31}{63}$, C's part. Then, as $\frac{31}{63} : \frac{2}{9}$, or as $31 : 14 ::$ £2000 : £$903\frac{7}{31}$; the part paid by A; and as $\frac{31}{63} : \frac{2}{7}$, or as $31 : 18 ::$ £2000 : £$1161\frac{9}{31}$, the part paid by B.

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* These questions are selected from various authors. Their solutions are annexed for the purpose of showing the more advanced student, how they and similar arithmetical questions may be resolved. They will be found useful in preparing the pupil for working the Miscellaneous Questions that follow them, and for resolving any of the more difficult questions that may be proposed for solution by common arithmetic.

8. Required the stocks of A and B, A's gain being £160, and B's £130, and A's stock £175 more than B's.

Sol. £160 − £130 = £30. Then, as £30 : £175 :: £160 : £933$\frac{1}{3}$, A's stock; and as £30 : £175 :: £130 : £758$\frac{1}{3}$, B's stock.

9. Given A's stock = £240, B's = £210; the whole gain = £120, and C's share of it = £30. Required A's and B's gains, and C's stock.

Sol. £120 − £30 = £90, the sum of A's and B's gains; and £240 + £210 = £450, the sum of their stocks. Then, as £450 : £90, or 5 : 1 :: £240 : £48, A's gain; as 5 : 1 :: £210 : £42, B's gain; and as 1 : 5 :: £30 : £150, C's stock.

10. If A gain £12 in 6 months, B £15 in 5 months, and C £2, in 9 months; what is the whole stock, C's part of it being £40?

Sol. In compound fellowship the gains are proportional to the *products* of the stocks and times; and, conversely, the stocks are proportional to the *quotients* obtained by dividing the gains by the times. Hence, as $\frac{21}{9}$: $\frac{12}{6}+\frac{15}{5}+\frac{21}{9}$ :: £40 : £125$\frac{5}{7}$, the whole stock. A's stock would be found thus: as $\frac{21}{9}$: $\frac{12}{6}$, or as 2$\frac{1}{3}$: 2, or as 7 : 6 :: £40 : £34$\frac{2}{7}$.

11. If A and B gain £13 - 10, B and C £12 - 12, and A and C £11 - 16 - 6; what is the gain of each?

Sol. From £18 - 19 - 3, half the sum of the given moneys, which is evidently equal to the gains of all the three, take £12 - 12, and there will remain £6 - 7 - 3, A's part. In like manner, £18 - 19 - 3 − £11 - 16 - 6 = £7 - 2 - 9, B's share; and £18 - 19 - 3 − £13 - 10 = £5 - 9 - 3, C's share.

12. If A can do a piece of work in 10 days, and B in 13; in what time will both do it, working at the same rate?

Sol. In one day A does $\frac{1}{10}$ and B $\frac{1}{13}$ of the work; therefore both together do in one day $\frac{1}{10}+\frac{1}{13}$ or $\frac{23}{130}$ of it. Hence, as $\frac{23}{130}$ of the work is to 1, the whole work; or, as 23 : 130 :: 1 day : 5$\frac{15}{23}$ days, the time required. It appears from this work, that the answer will be found by dividing the product of 10 and 13 by their sum.

13. A, B, and C can trench a field in 12 days; B, C, and D, in 14 days; C, D, and A, in 15 days; and D, A, and B, in 18 days. In what time would it be done by all of them together, and by each of them singly?

Sol. In one day, A, B, and C will do $\frac{1}{12}$ of the whole work; B, C, and D, $\frac{1}{14}$; C, D, and A, $\frac{1}{15}$; and D, A, and B, $\frac{1}{18}$. Let these be added together, and the sum, $\frac{349}{1260}$, is the part done by all working 3 days, since in each of the three parts, $\frac{1}{12}$, $\frac{1}{15}$, and $\frac{1}{18}$, of the whole work, there is one day's work of A; in each of the three parts, $\frac{1}{12}$, $\frac{1}{14}$, and $\frac{1}{18}$, one day's work of B; &c. Dividing the sum by 3, therefore, we have $\frac{349}{3780}$, the part done by all four in one day. Hence, as $\frac{349}{3780}$ of the work is to 1, the whole work; or as 349 : 3780 :: 1 day : 10$\frac{290}{349}$ days, the time in which it would be performed by all of them working together. Now, from $\frac{349}{3780}$, the part done in one day by A, B, C, and D, take $\frac{1}{14}$, the part done by B, C, and D, and the remainder $\frac{79}{3780}$, is the part done by A. Then, as $\frac{79}{3780}$: 1, or as 79

: 3780 :: 1 day : $47\frac{67}{79}$ days, the time in which A alone would perform the work. By proceeding in the same manner, we should find, that B would perform it in $38\frac{94}{97}$ days; C in $27\frac{27}{139}$ days; and D in $111\frac{3}{17}$ days.

14. A and B, at the opposite extremities of a wood, 135 fathoms in compass, begin to go round it, in the same direction, at the same time; A at the rate of 11 fathoms in 2 minutes, and B of 17 fathoms in 3 minutes. How many rounds will each make, before the one will overtake the other?

Sol. As 2 minutes : 3 minutes :: 11 fathoms : $16\frac{1}{2}$ fathoms, the space gone by A in three minutes. Hence it appears, that B, in going 17 fathoms, gains $\frac{1}{2}$ fathom on A; and the object of the question being to find how many rounds he will make in gaining half a round, we have this analogy; as $\frac{1}{2}$ fathom : 17 fathoms :: $\frac{1}{2}$ round : 17 rounds, the space to be gone over by B: consequently A will make $16\frac{1}{2}$ rounds.

15. Suppose A, B, and C, to start from the same point, and to travel in the same direction, round an island 73 miles in compass, A at the rate of 6, B of 10, and C of 16 miles per day: in what time will they be next together?

Sol. Since B gains 4 miles each day on A, as 4 miles : 73 miles :: 1 day : $18\frac{1}{4}$ days, the time in which B would gain a round on A, or in which these two would first be together again. Also, as 6 miles, the space gained each day by C on B : 73 miles :: 1 day : $12\frac{1}{6}$ days, the time in which B and C will first be together. Now the least common multiple of $18\frac{1}{4}$ and $12\frac{1}{6}$ (which are both divisible by $6\frac{1}{12}$), is readily found to be $36\frac{1}{2}$, the number of days required.

16. A person remarked that when he counted over his basket of nuts, two by two, three by three, four by four, five by five, or six by six, there was one remaining; but when he counted them by sevens there was no remainder. How many had he?

Sol. The least common multiple of 2, 3, 4, 5, and 6 being 60, it is evident, that if 61 were divisible by 7, it would answer the conditions of the question. This not being the case, however, let $60 \times 2 + 1$, $60 \times 3 + 1$, $60 \times 4 + 1$, &c., be tried successively, and it will be found that $301 = 63 \times 5 + 1$, is divisible by 7; and consequently this number answers the conditions of the question. If to this we add 420, the least common multiple of 2, 3, 4, 5, 6, and 7, the sum, 721, will be another answer; and by adding perpetually 420, we may find as many answers as we please.

17. At what time, between twelve and one o'clock, do the hour and minute hands of a common clock or watch, point in directions exactly opposite?

Sol. This is the same as to find in what time after twelve, at which time the hands are together, the minute hand will have gained half a round on the hour hand. Now it is evident, that in 12 hours the minute hand gains eleven rounds; and consequently one round is gained in the eleventh part of 12 hours, and half a round in half that time, or the eleventh part of 6 hours; that is, in $32\frac{8}{11}$ minutes. The time required, therefore, is $32\frac{8}{11}$ minutes after twelve o'clock.

18. If one ship, containing 150 hogsheads of wine, pay for toll at the Sound, the value of 2 hogsheads, wanting £6; and another, containing 240 hogsheads, pay at the same rate, the value of 2 hogsheads, and £18 besides; what is the value of the wine per hogshead?

Sol. The tolls must evidently be in the ratio of 150 to 240, or of 5 to 8. Hence the value of 2 hogsheads less by £6, must be to the value of 2 hogsheads together with £18, or, the half of each being taken, the value of 1 hogshead wanting £3, must be to the value of 1 hogshead together with £9, as 5 to 8. $\frac{5}{8}$ of the value of 1 hogshead, therefore, with $\frac{5}{8}$ of £9 must be equal to the value of 1 hogshead wanting £3; that is, $\frac{5}{8}$ of the value of 1 hogshead, with £$5\frac{5}{8}$, must be equal to the value of 1 hogshead wanting £3. Hence, £$5\frac{5}{8}$, must be equal to $\frac{3}{8}$ of the value of 1 hogshead wanting £3: and consequently $\frac{3}{8}$ of the value of a hogshead must be equal to £$8\frac{5}{8}$. Hence, as $\frac{3}{8}$ hhd. : 1 hhd. :: £$8\frac{5}{8}$: £23, the value of 1 hogshead.

19. If 3 men, or 4 women, can do a piece of work in 56 days, in what time will one man and one woman together, perform it?

Sol. In 56 days, one man will do $\frac{1}{3}$, and one woman $\frac{1}{4}$ of the work; and consequently, in the same time, one man and one woman will do $\frac{1}{3}+\frac{1}{4}$, or $\frac{7}{12}$, of the work. Hence, as $\frac{7}{12}$ of the work : 1, the whole work, :: 56 days : 96 days, the time required.

20. If 7 gallons of brandy cost as much as 9 gallons of rum, and 9 gallons of rum as much as 12 gallons of geneva, and the price of 3 gallons of these, taking 1 of each kind, was £2 - 2 - 6; what was the value of each per gallon?

Sol. It appears from the question, that the prices of 1 gallon of brandy, 1 of rum, and 1 of geneva are as $\frac{1}{7}$, $\frac{1}{9}$, and $\frac{1}{12}$; whence, by reducing these fractions to equivalent ones having a common denominator, and using the numerators, we find, that the prices are as 36, 28, and 21. We have, therefore, by the method of dividing into proportional parts, the following analogies: as 36 + 28 + 21, that is, as 85 : 36 :: £2 - 2 - 6 : 18*s.*, the value of the brandy per gallon; as 85 : 28 :: £2 - 2 - 6 : 14*s.*, that of the rum; and as 85 : 21 :: £2 - 2 - 6 : 10*s.* 6*d.*, that of the geneva.

MISCELLANEOUS QUESTIONS.*

1. If a person gain $8\frac{1}{3}$ per cent. by selling apples at the rate of 8 for $6\frac{1}{2}d.$, how much does he gain per cent. by selling them at the rate of 3 for $2\frac{1}{2}d.$? *Answ.* $11\frac{1}{9}$.

* The questions contained in this article are intended to exercise the advanced student in the use of the several rules and modes of operation, exhibited in the preceding part of this work. They are not adapted for the majority of arithmetical pupils; as for them they are too difficult, and possess too little practical utility. It is hoped, however, that, besides affording much practice in calculation, and in the application of the rules already delivered, they will form useful exercises for the reasoning powers of those who have taste or ability for such speculations. The great and principal object with every teacher of arithmetic, should be, to make his pupils acquire an extensive and substantial *practical* knowledge of this science, without occupying their time and attention with puzzling or difficult questions. At the same time, when he meets with pupils of capacity, and of considerable proficiency, it may be very proper to direct their

2. If eggs be bought at the rate of 5 for a penny, how must they be sold to gain 40 per cent.? *Answ.* At the rate of 25 for 7*d.*

3. If 150 apples cost 9*s.* $4\frac{1}{2}$*d.*, how many of them must be sold at the rate of 8 for $6\frac{1}{2}$*d.*, and how many at the rate of 3 for $2\frac{1}{2}$*d.*, that the gain on the whole may be 10 per cent.? *Answ.* 90 at 3 for $2\frac{1}{2}$*d.*, and 60 at 8 for $6\frac{1}{2}$*d.*

4. A merchant engages a clerk at the rate of £20 for the first year, £25 for the second, £30 for the third, &c., thus augmenting his salary by £5 each year. How long must the clerk retain his situation, so as to receive on the whole as much as he would have received, had his salary been fixed at £52 - 10 per annum? *Answ.* 14 years.

5. Three gentlemen contribute £164 - 5 towards the building of a church at the distance of 2 miles from the first, $2\frac{7}{8}$ miles from the second, and $3\frac{1}{2}$ miles from the third; and they agree that their shares shall be reciprocally proportional to their distances from the church. How much must they severally contribute? *Answ.* £72 - 9, £50 - 8, and £41 - 8.

6. A hosier sells 90 pair of stockings and gloves for £12 - 10, the stockings at 3*s.*, and the gloves at 2*s.* 6*d.* per pair. Required the number of each. *Answ.* 50 pair of stockings, and 40 pair of gloves.

7. A son having asked his father's age, the father replied: "Your age is twelve years; to which if five-eighths of both our ages be added, the sum will be equal to mine." What was the father's age? *Answ.* 52 years.

8. Three merchants having formed a joint stock of £1064, A's stock continues in trade 5 months, B's 8 months, C's 12 months; and A's share of the gain is £114, B's £133 - 4, and C's £165. What was the stock of each? *Answ.* A's £456, B's £333, and C's £275.

9. The stocks of three partners, X, Y, and Z, continue in trade 8, 10, and 7 months respectively; and their respective gains are £115 - 10, £204 - 15, and £183 - 15. Hence, it is required to find their several stocks, the difference between those of Y and Z being £220. *Answ.* X's stock £550, Y's £780, and Z's £1000.

10. The joint sum of two series of continual proportionals, consisting of five terms each, and having a common mean, is $80\frac{77}{90}$, and their ratios are $1\frac{1}{2}$ and $2\frac{1}{2}$. Required the series. *Answ.* $2\frac{2}{9}$, $3\frac{1}{3}$, 5, $7\frac{1}{2}$, $11\frac{1}{4}$; and $\frac{4}{5}$, 2, 5, $12\frac{1}{2}$, $31\frac{1}{4}$.

11. If A, B, and C could pave a street in 18 days; B, C, and D in 20 days; C, D, and A, in 24 days; and D, A, and B, in 27 days; in what times would it be done by all of them together, and by each of them singly? *Answ.* By all in $16\frac{56}{199}$ days; by A in $87\frac{21}{37}$ days; by B in $50\frac{5}{8}$ days; by C in $41\frac{1}{79}$ days; and by D in $170\frac{10}{19}$ days.

12. If A could reap a field in 13 days, and B in 16 days, in what time would both together reap it? *Answ.* In $7\frac{5}{29}$ days.

13. If A and B, with C working half time, could build a wall in 21 days; B and C, with D working half time, in 24 days; C and D, with A working half time, in 28 days; and D and A, with B working

attention to such questions as are contained in this article. By this means he will have a farther proof of their capacity, and he may lay the foundation of future proficiency in other departments of mathematical science.

half time, in 32 days; in what times would it be built by all of them together, and by each of them singly? *Answ.* A would finish it in $52\frac{1}{2}$ days; B in $57\frac{27}{29}$ days; C in $44\frac{4}{19}$ days; D in 280 days; and all in 16 days.

14. The stocks of three partners, A, B, and C, are £350, £220, and £250, and their gains £112, £88, and £120, respectively; and B's stock continued in trade 2 months longer than A's. Required the time the money of each continued in trade. *Answ.* 8, 10, and 12 months, respectively.

15. In what arithmetical scale would five hundred and fifty-four be expressed by 95? *Answ.* In the scale whose radix is 61.

16. If a gallon of water were resolved into the oxygen and hydrogen of which it is composed, it is required to determine the bulk into which it would thus be expanded, water being 709 times heavier than an equal bulk of oxygen, and 11315 times heavier than an equal bulk of hydrogen. *Answ.* The two gases would fill $1890\frac{1}{3}$ gallons, nearly.

17. The Napierian logarithm* of any number, is to the common logarithm of the same number as 1 is to ·43429448, nearly. Required the series of ratios converging to this ratio. *Answ.* 2 to 1; 7 to 3; 23 to 10; 76 to 33; 99 to 43; 175 to 76 (or 700 to 304); 624 to 271; 3919 to 1702; 12381 to 5377; 16300 to 7079, &c.

18. By reducing the fraction whose numerator is the square root of 5, and denominator the square root of 11, to a continued fraction, find the first seven of the series of fractions converging to its value. *Answ.* $\frac{1}{1}$, $\frac{2}{3}$, $\frac{29}{43}$, $\frac{60}{89}$, $\frac{149}{221}$, $\frac{358}{531}$, and $\frac{5161}{7655}$.

19. A person, in discounting a bill, at 6 per cent. per annum, according to the common or false method, finds that he has $6\frac{1}{2}$ per cent. per annum, for his money. How long must the bill have been discounted before it was due? *Answ.* 1 year and 103 days, nearly.

20. A person pays £54 for the insurance of goods at $3\frac{3}{4}$ per cent.; and he finds, that in case of the goods being lost, he will by this means be entitled to the value of the goods, the premium of insurance, and £5 besides. What is the value of the goods? *Answ.* £1381.

21. Reduce five ninths, to a fraction in the septenary scale of notation, whose denominator in that scale may be expressed by 1, with as many ciphers annexed, as there are figures in the numerator. *Answ.* The numerator will be ·361361361, &c., or $\cdot\dot{3}6\dot{1}$.

22. If a merchant each year increase his capital by a fifth part of itself, except an expenditure of £400 per annum, and at the end of 15 years be worth £12,000; what was his original capital? *Answ.* £2649 - 1 - 1.

23. A servant draws off a gallon, each day, for 20 days, from a cask containing 10 gallons of rum, each time supplying the deficiency by the addition of a gallon of water: and then, to escape detection, he again draws off 20 gallons, supplying the deficiency each time by a gallon of rum. How much water still remains in the cask? *Answ.* 1·0679577 gallon, or rather more than a gallon and half a pint.

* Napierian logarithms are also frequently, but improperly, called *hyperbolic* logarithms.—The number 0·4342944812, &c., is called the *modulus* of the system of common logarithms.

24. A sells a quantity of tea, which cost him £246 - 12, to B; and B sells it to C, who disposes of it for £391 - 11 - 10. Required the prices at which A and B sold it, each of the three merchants having gained at the same rate per cent. *Answ.* A sold it for £287 - 14, and B for £335 - 13.

25. A and B set out from the same place, and in the same direction. A travels uniformly 18 miles per day, and after 9 days turns and goes back as far as B has travelled during those 9 days; he then turns again, and pursuing his journey, overtakes B $22\frac{1}{2}$ days after the time they first set out. It is required to find the rate at which B uniformly travelled. *Answ.* 10 miles per day.

26. A merchant every year gains 50 per cent. on his capital, of which he spends £300 per annum in house and other expenses; and at the end of 4 years he finds himself possessed of a capital four times as great as what he had at commencing business. What was his original capital? *Answ.* £$2294\frac{2}{17}$.

27. At what time does the sun set, when the length of the day (from sunrise till sunset) is four times the length of the morning or evening twilight, and the evening twilight two sevenths of the time from its termination till day-break? *Answ.* At $3\frac{3}{19}$ minutes past 5 o'clock.

28. How will 13579 in the trigesimal scale, be expressed in the duodecimal scale. *Answ.* 372433.

29. Find the first nine fractions approaching to the ratio of 1 to the cube root of 2. *Answ.* $\frac{1}{1}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{23}{29}$, $\frac{27}{34}$, $\frac{50}{63}$, $\frac{227}{286}$, $\frac{277}{349}$, and $\frac{504}{635}$.

30. How will that fraction be expressed in the decimal scale, whose denominator in the octary scale is a unit with as many ciphers annexed as there are figures in its numerator, and its numerator 644 repeated without end? *Answ.* $\frac{60}{73}$.

31. What is the difference between $\frac{4}{12}$ in the quinary scale, and $\frac{7}{31}$ in the nonary scale? *Answ.* $\frac{27}{133}$ in the decimal notation.

32. What is the product of $\frac{54}{7}$ in the duodecimal scale, and $\frac{7}{54}$ in the octary scale? *Answ.* $1\frac{5}{11}$ in the decimal scale.

33. It is required to find a sum of money, of which, in the space of 4 years, the *true* discount, at simple interest, is £5 more at the rate of 6 than of 4 per cent. per annum. *Answ.* £89 - 18.

34. One third of a quantity of flour being sold to gain a certain rate per cent., one fourth to gain twice that rate, and the remainder to gain three times the same rate, it is required to find the gain per cent. on each part, the gain upon the whole being 20 per cent. *Answ.* The gains per cent. are $9\frac{3}{5}$, $19\frac{1}{5}$, and $28\frac{4}{5}$.

35. A man travels from his own house to Belfast in 4 days, and home again in 5 days, travelling each day, during the whole journey, one mile less than he did the preceding. How far does he live from Belfast? *Answ.* 90 miles.

36. What is the radix of the arithmetical scale of notation, in which 9(20)(12)609 in the trigesimal notation will be expressed by 5000004? *Answ.* 19.

37. The men employed by a farmer, work 12 hours, the women 9 hours, and the boys 8 hours, each day: for labouring the same number of hours, each man receives a half more than each woman,

and each woman a third more than each boy: the entire sum paid to all the women each day is double of the sum paid to all the boys; and for every five shillings earned by all the women each day, twelve shillings are earned by all the men. Hence it is required to find the number of each class employed, the entire number being 59. *Answ.* 24 men, 20 women, and 15 boys.

38. A man leaves to his eldest child one fourth of his property; to his second, one fourth of the remainder, and £350 besides; to his third, one fourth of the remainder and £975; to his youngest, one fourth of the remainder and £1400; and what still remains he bequeaths to his wife, whose share is found to be one fifth of the whole. Hence it is required to find the value of the whole property. *Answ.* £20,000.

39. The less of two bales of cloth is bought at the rate of twice as many pence per yard as it contains yards, and costs £31 - 0 - 2 more than the greater, which contains 4 yards for every 3 in the less, and is bought at the rate of as many pence per yard as it contains yards. How many yards are contained in each? *Answ.* 244 yards in the greater, and 183 in the less.

40. It is required to find a sum of money such that its true discount, for one year, at 5 per cent., will be £1 more than the sum of the true discounts of one half of it at 4 per cent., and the rest at 6 per cent. *Answ.* £11,575 - 4.

41. A property of £10,000 is left to 4 children whose ages are 6, 8, 10, and 12 years respectively; and it is so divided among them, that their several shares being improved at $4\frac{1}{2}$ per cent. per annum, compound interest, they shall all have equal properties at the age of 21. What is the sum left to each? *Answ.* £2180 - 3 - $4\frac{1}{4}$, £2380 - 15 - $11\frac{1}{4}$, £2599 - 17 - $9\frac{3}{4}$, and £2839 - 2 - $10\frac{3}{4}$.

42. A property is left to four children, one aged 6 years, two aged 9 years each, and one aged 11 years, in such a manner that all their properties are to be equal on their coming to age, compound interest being allowed at 4 per cent. per annum. Now, one of the twins dying before the period at which the eldest would be of age, his share is divided among the rest in such a manner that their properties may be still equal, when they come to age; and, in consequence, each is found at that period to have £1000 more than he would otherwise have had. What was the value of the entire property bequeathed? *Answ.* £7367 - 2 - $11\frac{1}{4}$.

43. A man owes a debt, to be paid in four equal instalments at the end of 4, 9, 12, and 20 months respectively; and he finds, that discount being allowed, according to the true method, at 5 per cent. per annum, £750 paid at present will discharge the whole debt. How much did he owe? *Answ.* £$784\frac{978194}{1321009}$.

44. If a merchant commence trade with a capital of £5000, and gain so much, that, after paying all expenses, his capital, each year, is increased by a tenth part of itself wanting £100, how much will he be worth at the end of 20 years? *Answ.* £27910.

45. A man borrowed £500, and agreed to pay simple interest at 5 per cent. per annum. At the end of 11 months, he paid one part of the principal with its interest; 8 month after, he paid another

part with its interest from the time it was borrowed; and 11 months after that, he paid the remainder of the principal with its interest, in like manner, from the time it was borrowed. What was the amount of each payment, each of the last two being double of the first? *Answ.* The first payment, £$108\frac{669717}{670976}$, and each of the others, £$217\frac{334229}{335488}$.

46. Mercury revolves round the sun in 87 days, 23 hours, 15 minutes, 44 seconds, and the earth in 365 days, 6 hours, 9 minutes, 12 seconds. Required the first seven approximate ratios of these periods. *Answ.* $\frac{1}{4}$, $\frac{6}{25}$, $\frac{7}{29}$, $\frac{13}{54}$, $\frac{33}{137}$, $\frac{46}{191}$, and $\frac{217}{901}$.

47. Venus revolves round the sun in 224 days, 16 hours, 49 minutes, 11 seconds, and the earth in the time stated in the preceding question. Required the first eight fractions approaching to the ratios of the periods. *Answ.* $\frac{1}{1}$, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{5}$, $\frac{8}{13}$, $\frac{235}{382}$, $\frac{478}{777}$, and $\frac{14575}{23692}$.

48. 35 lbs. of tea being mixed with 20 lbs. of a better quality, the mixture is found to be worth 7*s.* 4*d.* per lb. Required the value of each kind, the difference of their values being 1*s.* 10*d.* per lb. *Answ.* 8*s.* 6*d.* and 6*s.* 8*d.* per lb.

49. If a person lend £7000 at 6 per cent. per annum, compound interest, and allow the interest to accumulate in the hands of the creditor, except £240 per annum, which he lifts for family expenditure; how much will the creditor owe him at the end of 16 years? *Answ.* £11,621 - 1 - $1\frac{1}{4}$.

50. If a boy read each day 2 lines more of Virgil than he did the day before, and find, that having read a certain quantity in 18 days, he will read, at this rate of increase, the same quantity in the next 14 days; how much will he read in the whole time? *Answ.* 4032 lines.

51. Two men, A and B, are on a straight road, on the opposite sides of a gate, and distant from it 308 yards and 277 yards respectively, and travel each towards the original station of the other. How long must they walk till their distances from the gate will be equal, B travelling 2 yards, and A $2\frac{1}{3}$ yards, per second? *Answ.* 1 minute, 33 seconds, or 2 minutes, 15 seconds.

52. Every thing being supposed to be as in the preceding question, at what time will each be at the same distance from the original station of the other, as the other is from his? *Answ.* In $4\frac{1}{3}$ minutes after starting.

53. If a person borrow £1100 at 6 per cent. per annum, compound interest, and agree to pay both principal and interest in eleven equal annual payments, how much must each payment be, the first being made at the end of the first year? *Answ.* £139 - 9 - $5\frac{1}{4}$.

54. If a farm of 84 acres be held at £1 - 7 - 6 per acre, on a lease of which 48 years are unexpired, what fine must be paid at present to reduce the rent to 10*s.* per acre during the last 30 years of the lease, compound interest being allowed at 6 per cent. per annum? *Answ.* £354 - 8 - $11\frac{1}{2}$.

55. A man aged 45 has a pension of £300 a year during his own life; but he wishes to exchange it for another to continue not only during his own life, but also during that of his wife, aged 40. What will this pension be, money being supposed to be improvable at 6 per cent. per annum, compound interest? *Answ.* £236 - 14 - $7\frac{1}{2}$.

56. Required the converging fractions approaching to the ratio of 5 hours, 48 minutes, 48 seconds, and 24 hours. *Answ.* $\frac{1}{4}$, $\frac{7}{29}$, $\frac{8}{33}$, $\frac{31}{128}$, and $\frac{39}{165}$.

57. If the acting partner in a mercantile concern contribute £1000 to the original joint stock of the company, and annually increase this sum by £150 saved from his salary; to how much will his share of the joint stock amount, at the end of 11 years, on the supposition, that, after all expenses are paid, there is a clear gain of 10 per cent. per annum on the entire capital? *Answ.* £5632 - 15 - 10.

58. Suppose 17 gallons of spirits, at 10*s.* 6*d.* per gallon, to be mixed with 7 gallons at a different price. What was the price of the latter per gallon, if 20 per cent. be gained by selling the mixture at 13*s.* per gallon? *Answ.* 11*s.* $7\frac{5}{7}$*d.*

59. A merchant bequeaths £1000 among six clerks in proportion to their salaries, and the periods they have held their situations. Now, one of them has held his situation five years, and his salary is £120; two of them four years, with salaries of £75 each; and the rest two years, with salaries of £60 each. Required their several shares. *Answ.* The share of the first £384 - 12 - $3\frac{9}{13}$; of the next two £192 - 6 - $1\frac{11}{13}$ each; and of the rest £76 - 18 - $5\frac{7}{13}$ each.

60. If a shopkeeper each year double his capital, except an expenditure of £240 per annum, and, at the end of four years, be worth only three fourths of his original capital; what had he at commencing trade? *Answ.* £236 - 1 - $3\frac{45}{61}$.

61. What sum will amount to £1 more at simple than at compound interest, in 4 months, at 5 per cent. per annum? *Answ.* £3699 - 9 - 2.

62. Required the sum of the infinite series, $\frac{2}{5}-\frac{4}{15}+\frac{8}{45}-\frac{16}{135}+\frac{32}{405}-$ &c. *Answ.* $\frac{6}{25}$.

63. If a bank borrow £10,000 at 4, and employ it in discounting bills, at 6 per cent. per annum, and afterwards borrow money at 2, and discount bills with it, at 4 per cent. per annum; how much must be borrowed, and so employed, in the latter case, that the gain may be the same as in the former, the bills being discounted in both cases six months before they would be due? *Answ.* £$10{,}499\frac{1447}{4947}$.

64. Reduce $\frac{1}{2}\sqrt{5}-\frac{1}{2}$ and $\frac{3}{2}-\frac{1}{2}\sqrt{5}$* to continued fractions, and find the fractions, converging to the value of each. *Answ.* The continued fractions are—

$$\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\&c.}}}}}; \quad \text{and} \quad \frac{1}{2+\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1}}}}}\ \&c.:$$

and the converging fractions are $\frac{1}{1}$, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{5}$, $\frac{5}{8}$, $\frac{8}{13}$, $\frac{13}{21}$, &c., and $\frac{1}{2}$, $\frac{1}{3}$, $\frac{2}{5}$, $\frac{3}{8}$, $\frac{5}{13}$, $\frac{8}{21}$, &c.

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* The mathematical reader will perceive that these expressions are the parts of the number 1, when divided in extreme and mean ratio; that is, so that the whole is to the greater part, as the greater to the less.

MENTAL ARITHMETIC.*

In addition and subtraction, the pupil should perform, mentally, a great number of exercises such as the following:—

Add together 15 and 18; 23 and 35; 82 and 94; 25, 41, and 17; 318, 202, and 156, &c.: also 5*s*. 8*d*. and 7*s*. 4*d*.; £1 - 17 - 9 and 12*s*. 10*d*.; 2*s*. 4½*d*. and 3*s*. 9*d*., &c.

Find the difference of 40 and 23; of 82 and 45; of 133 and 98; of 212 and 154, &c.: also of £1 and 4*s*. 2*d*.; of 17*s*. 3*d*. and 6*s*. 6*d*.; of £1 - 12 - 0 and 15*s*. 4*d*., &c.

From the sum of 17 and 52, take 35; from 365 take the sum of 73 and 91; from the sum of 12*s*. 8*d*. and 11*s*. 8*d*. take 15*s*. 6*d*., &c.

In adding, it is often of advantage to increase one or more of the numbers, and to diminish one or more others equally, so that some of the numbers may be exchanged for others more simple and manageable. Thus, the sum of 87 and 55 is evidently the same as that of 90 and 52, or 142: and the sum of 197, 301, and 69 is plainly the same as that of 200, 300, and 67, or 567.

In subtracting, it frequently gives facility, if we increase both the numbers equally. Thus, to subtract 94 from 141, we may add 6 to each, and then 100 is to be taken from 147, which gives for difference 47. So likewise, to find the difference of 59 and 81, we may take 60 from 82.

Quantities of different denominations, particularly money, may often be added or subtracted very easily in the same manner. Thus, the sum of 11*s*. 10*d*. and 3*s*. 9*d*. is the same as that of 12*s*. 0*d*. and 3*s*. 7*d*.; and the sum of £7 - 16 - 0 and £4 - 8 - 0, is the same as that of £8 and £4 - 4 - 0: while the difference of £11 - 6 - 0 and £5 - 17 - 0 is the same as that of £11 - 9 - 0 and £6.

In multiplication, many questions, such as the following, ought to be performed mentally: 7 times 48,—6 times 74,—9 times 86,—12 times 112,—4 times £1 - 6 - 8,—7 times 2*s*. 6*d*.,—8 times 10*s*. 10*d*. —9 times 10½*d*.,—5 times £2 - 12 - 6, &c.: also 23 times 32,—41 times 104,—24 times 61, &c. In exercises of the latter kind, the learner may either proceed in the common way, conceiving that he writes down the partial products, and adds them together;

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* Many operations in arithmetic may readily be performed mentally, without the use of pen or pencil. By most persons this can be done only when the processes are short and simple, the more complicated operations being too laborious. It is often of considerable use, in a practical point of view, to be able to work questions in this way; and the mind is strengthened by the exertions required in the process. In the following short article on the subject, various examples are given, and various hints thrown out, which, it is hoped, will be found useful.

or, should he find that fatiguing, he may separate the multiplier into convenient parts, multiply by them separately, and add the amounts. Thus, to find how many weeks there are in 24 years, we may multiply 52 by 20 and 4 separately, and add together the products 1040 and 208.

To multiply 42 by 35, we may multiply 21 by 70, halving one of the factors and doubling the other. In a similar manner, we may multiply by 15, 45, and 55, doubling each, and either halving the other factor at first, or retaining it, and halving the final result.

In multiplying 97 by 28, we get 28 times 100 wanting 28 times 3, or 2800 wanting 84, that is, 2716. Many other simplifications will readily occur to persons who shall turn their attention to the subject.

In division, let the pupil work, mentally, many questions, such as the following: Divide 177 by 3, 291 by 7, 364 by 8, 583 by 11, £4 by 9, 11*s*. 6*d*. by 4, £1 - 5 - 9 by 7, £17 - 10 - 0 by 8, &c.

To divide by 15, 35, 45, or 55, double the dividend, and divide by 30, 70, 90, or 110, halving the remainder, if there be any.

By combining these processes, we may work various questions, such as the following: Since one seventh of 84 is 12, two sevenths of it are twice 12, or 24; three sevenths are three times 12, or 36, &c. Since one eleventh of 77 is 7, five elevenths of it are five times 7, or 35; ten elevenths of it, ten times 7, or 70, &c. Conversely, to find of what number 18 is three fifths, by dividing 18 by 3, we find that one fifth of the number is 6; five times 6, therefore, or 30, is the number. So also, to find how many yards of ribbon, at 5*d*. per yard, may be bought for 13*s*. 9*d*., reduce, mentally, 13*s*. 9*d*. to pence; then dividing the result, 165, by 5 (or doubling it and rejecting the last figure), we get 33, the number of yards.

One of the commonest and most useful applications of mental arithmetic is in the computation of prices. It can be readily employed in this way, when the number of articles is small, not exceeding 10 or 12. Thus, any one can find in an instant, by multiplication, that 4 lbs. of sugar, at $7\frac{1}{2}d$ per lb., amount to 2*s*. 6*d*.; that 10 yards of carpeting, at 3*s*. 6*d*. per yard, cost £1 - 15 - 0; that the price of 11 yards of cloth, at 8*s*. 4*d*. per yard, is £4 - 11 - 8, &c.

In some cases, also, the price may be readily found, when the number of articles is considerable, especially if it bear some peculiar and simple relation to the divisions of money. Thus, a dozen of articles, at a penny each, cost a shilling; at 2 pence, 2 shillings; at 7 pence, 7 shillings; at 4 pence halfpenny, 4*s*. 6*d*.; at 8 pence three farthings, 8*s*. 9*d*., &c.

RULE I. *To find the price of any number of dozens*, regard the pence in the price as shillings, and multiply by the number of dozens.

Thus, to find the price of 72 (6 dozen) at 11*d*., we multiply 11*s*. by 6; and to find the price of 108 (9 dozen) at $8\frac{3}{4}d$., we multiply 8*s*. 9*d*. by 9.

Rule II. *When the number of articles does not consist of exact dozens,* find the price of the dozens by the last rule, and increase or diminish it, as the case may require, by the price of the other articles.

Thus, to find the price of 86 (7 dozen and 2), at 1*s.* 4*d.* each, we multiply 16*s.*, the price of one dozen, by 7, and to the product, £5 - 12 - 0, we add 2*s.* 8*d.*, the price of 2. Again, to find the price of 59 (5 dozen wanting 1) at $10\frac{1}{2}$*d.* each, we multiply 10*s.* 6*d.* by 5, and by taking $10\frac{1}{2}$*d.* from the product, £2 - 12 - 6, we get £2 - 11 - $7\frac{1}{2}$, the required price.

Since 20 articles, at a shilling each, cost a pound, we have the following rule:—

Rule III. *To find the price of* 20 *articles,* the shillings in the price are to be taken as pounds: and if there be pence, they are to be regarded as so many twelfths of a pound; a penny being thus to be taken as 1*s.* 8*d.*, 2*d.* as 3*s.* 4*d.*, 3*d.* as 5*s.* 0*d.*, 6*d.* as 10*s.* 0*d.*, 9*d.* as 15*s.* 0*d.*, &c.

Thus, 20 articles, at 14*s.* 0*d.* each, will cost £14; at 12*s.* 9*d.*, £12 - 15 - 0; at 2*s.* 4*d.*, £2 - 6 - 8, &c. Also, since 80 is four times 20, the price of 80 at 5*s.* 8*d.* is 4 times £5 - 13 - 4, or £22 - 13 - 4, and the price of 102 at 16*s.* 0*d.* is 5 times £16, together with twice 16*s.* 0*d.*; that is, £81 - 12 - 0.

Since, at a penny per ℔., a cwt. costs 112 pence, or 9*s.* 4*d.*, at 2*d.* it will cost twice as much; at 3*d.*, three times as much, &c. Hence—

Rule IV. *From the price of a pound to find the price of a hundred weight,* multiply 9*s.* 4*d.* by the pence in the given price.

Thus, at 5*d.* per ℔, a cwt. will cost 5 times 9*s.* 4*d.*, or £2 - 6 - 8; while, at $7\frac{3}{4}$*d.*, it would cost 8 times 9*s.* 4*d.* (£3 - 14 - 8) wanting 2*s.* 4*d.* (one fourth of 9*s.* 4*d.*); the price therefore would be £3 - 12 - 4.

Rule V. *To find the price of* 120 *articles:* Regard the pence in the given price as pounds, and divide by 2.

Thus, at 4*d.* per ℔., 120 ℔s. of pork would cost £2, the half of £4; and at $7\frac{3}{4}$*d.* per ℔., the cost of 120 ℔s. would be the half of £7 - 15 - 0.

Rule VI. *The price of* 100 *articles* may be found by taking each shilling as £5, and consequently 6*d.* as £2 - 10 - 0, 3*d.* as £1 - 5 - 0, a penny as 8*s.* 4*d.*, &c.

Thus, 100 at 9*s.* 6*d.* would cost £47 - 10; 100 at 2*s.* 1*d.*, £10 - 8 - 4, &c.

RULE VII. *In finding the price of* 100 *articles*, if the price of each article be small, reduce it to farthings, and take those farthings as pence, and their double as shillings.

Thus, 100 at $1\frac{3}{4}d.$, or 7 farthings each, amount to twice 7 shillings, together with 7*d.*; that is, to 14*s.* 7*d.* The reason is plain, since, at a farthing each, 100 articles cost 2*s.* 1*d.*; that is, twice one shilling, together with one penny.

Sometimes the following rule may be employed with advantage:—

RULE VIII. Find the price of the given number of articles at one farthing, one penny, or one shilling, and multiply the result by the number of farthings, of pence, or of shillings in the given rate.

Thus, to find the price of 68 at $2\frac{3}{4}d.$, or 11 farthings, since 68 articles, at a farthing each, cost 1*s.* 5*d.*, by multiplying this by 11, we get 15*s.* 7*d.*, the required price. So, likewise, to find the amount of 37 yards of ribbon at 7*d.* per yard, we multiply 3*s.* 1*d.*, the price at one penny per yard, by 7; the product, £1 - 1 - 7, is the required price. Again, to find the price of 82 cwt. at 9 shillings, multiply £4 - 2 - 0 by 9.—The *reason* is evident.

Computations in interest and discount may sometimes be performed mentally with considerable facility, especially by means of contrivances which have been fallen upon to shorten the work in particular cases. The following are some of the more useful of these:—

RULE IX. *To find the interest of any number of pounds for a given number of months at* 5 *per cent. per annum:* Take the pounds as pence, and multiply by the months.

Thus, to find the interest of £34 for 7 months at 5 per cent. per annum, multiply 34*d.* or 2*s.* 10*d.* by 7, and the product 19*s.* 10*d.* is the answer. So likewise, the interest of £27 - 10, at the same rate, for 9 months, is 9 times $27\frac{1}{2}d.$, or 9 times 2*s.* $3\frac{1}{2}d.$; that is, £1 - 0 - $7\frac{1}{2}$. The *reason* is, that at 5 per cent. per annum the interest of a pound is one penny for one month, and consequently 2 pence for 2 months, 3 pence for 3 months, &c.

From the interest at 5 per cent. that at other rates may be easily derived. Thus, to find the interest of £115 for 2 months, at 4 per cent. per annum, we have twice 115 pence, or twice 9*s.* 7*d.*, that is, 19*s.* 2*d.*, the interest at 5 per cent.; from which taking 3*s.* 10*d.* (one fifth of itself), we obtain 15*s.* 4*d.*, the interest required. Had the rate been 6 per cent. we should have *added* one fifth; or we might employ the following method:—

RULE X. *To find interest for months at* 6 *per cent. per annum:* Multiply the principal by the months; increase

the unit figure by a fifth of itself, to find the pence of the answer, and take the others as expressing shillings.

Thus, to compute the interest of £36 for 8 months at 6 per cent. per annum, by multiplying 36 by 8, we get 288; a fifth of the last figure of which being nearly $1\frac{1}{2}$, we have for the required interest 28*s.* $9\frac{1}{2}$*d.*, or £1 - 8 - $9\frac{1}{2}$*d.* The *reason* will be plain from considering, that to find interest for months without abbreviation, we should divide the continual product of the principal, the rate, and the months by 1200, and that, when the rate is 6 per cent., we may omit the multiplication by 6, and divide by 200 (or by 10 and 20), instead of 1200. Now, the cutting off of one figure divides by 10, while taking the rest as shillings instead of pounds, divides by 20. The figure cut off is plainly tenths of a shilling, and the increasing of it by a fifth of itself gives twelfths of a shilling, or pence.

From the interest at 6 per cent. that at other rates may easily be found. Thus, to find the interest of £45 for 11 months at 4 per cent. per annum, by taking the product of 45 and 11, we get 495, which gives 49*s.* 6*d.*, or £2 - 9 - 6, for the interest at 6 per cent.; and taking from this a third of itself, we get £1 - 13 - 0, the interest required. Had the rate been 5 per cent., we should have subtracted a sixth; while for $4\frac{1}{2}$ per cent. we should have subtracted a fourth; for 3 per cent., taken a half, &c.

RULE XI. *To find interest for days at 5 per cent. per annum:* Multiply the principal by the days, and divide the product by 3, or multiply one of them by a third of the other: a tenth of the result is the answer in pence nearly. To correct it, reject a penny for every 6 shillings, or $3\frac{1}{4}$*d.* for every pound contained in it. Should this correction be considerable, add a penny for every 6 shillings in it to the answer.

To find, for example, the interest of £24 for 52 days at 5 per cent. per annum, we multiply 52 by 8 (a third of 24), and cutting off the last figure of 416, the product, we get 41·6*d.*, or 3*s.* $5\frac{1}{2}$*d.* nearly, and as this is nearly the half of 6 shillings, we reject a halfpenny for the correction, and find for answer 3*s.* 5*d.* So also, to find the interest of £80 for 80 days, at 4 per cent. per annum, by multiplying 80 by 80 and dividing by 3, we get 2133, the tenth of which, $213\frac{1}{4}$, taken as pence, is 17*s.* $9\frac{1}{4}$*d.* By rejecting 3*d.* from this, as correction, we get 17*s.* $6\frac{1}{4}$*d.*, the interest at 5 per cent. Lastly, we diminish this by 3*s.* 6*d.*, a fifth of itself, and we get 14*s.* $0\frac{1}{4}$*d.*, the answer.

As to the *reason* of this rule, by taking pounds as pence, we divide by 240, so that, upon the whole, we divide by $3 \times 10 \times 240$, or 7200. Now, the divisor (see page 230) should be 7300; and hence the necessity for the correction.

EXERCISES.

Add the following :—

Exer.

1. 28 and 34.
2. 95 and 75.
3. 288 and 312.
4. 27, 47, and 52.
5. 82, 99, and 112.
6. 15*s*. 6*d*. and 9*s*. 10*d*.
7. £2 - 18 and £5 - 8.
8. 2*s*. 10½*d*. and 7*s*. 3½*d*.
9. 16*s*. 2*d*., 14*s*. 11*d*., 3*s*. 9*d*.

Find the differences of the following :—

Exer.

10. 77 and 29.
11. 111 and 44.
12. 256 and 177.
13. 45 + 49 and 66.
14. 100 and 36 + 22.
15. 18*s*. 4*d*. and 12*s*. 6*d*.
16. £2 - 10 and £1 - 11 - 8.
17. 9*s*. 7*d*. and 3*s*. 9½*d*.

Multiply the following :—

Exer.

18. 77 by 3.
19. 124 by 4.
20. 183 by 5.
21. 88 by 7.
22. 128 by 8.
23. 99 by 9.
24. 125 by 11.
25. 93 by 12.
26. 23 by 36.
27. 45 by 28.
28. 96 by 26.
29. 89 by 35.
30. 72 by 55.
31. 47 by 47.
32. 74 by 98.
33. 12*s*. 8*d*. by 3.
34. £5 - 11 by 5.
35. 11*s*. 10*d*. by 6.
36. £1 - 3 - 6 by 8.
37. £3 - 6 - 8 by 9.
38. £4 - 5 - 4 by 10.
39. 10*s*. 7½*d*. by 12.

Divide the following :—

Exer.

40. 733 by 5.
41. 252 by 6.
42. 847 by 7.
43. 1010 by 9.
44. 237 by 12.
45. 745 by 15.
46. 575 by 25.
47. 1000 by 35.
48. £5 - 9 - 6 by 4.
49. £4 - 12 - 6 by 5.
50. £10 - 14 - 6 by 6.
51. £80 by 7.
52. £1 - 18 - 8 by 8.
53. £5 - 5 - 9 by 9.
54. £132 - 9 - 2 by 10.
55. £40 - 1 - 6 by 12.

EXERCISES ON RULE I.

Exercises.	*Answers.* £	s.	d.
56. 24 at 4*d.*	0	8	0
57. 48 at 7½*d.*	1	10	0
58. 72 at 1*s.* 10*d.*	6	12	0
59. 96 at 1*s.* 7*d.*	7	12	0
60. 132 at 9¼*d.*	5	1	9
61. 144 at 10¾*d.*	6	9	0

EXERCISES ON RULE II.

Exercises.	*Answers.* £	s.	d.
62. 73 at 7*d.*	2	2	7
63. 71 at 10*d.*	2	19	2
64. 106 at 1*s.* 1*d.*	5	14	10
65. 87 at 2*s.* 8*d.*	11	12	0
66. 57 at 1½*d.*	0	7	1½

EXERCISES ON RULE III.

Exercises.	*Answers.* £	s.	d.
67. 20 at 11*s.* 6*d.*	11	10	0
68. —— £2 - 7 - 3	47	5	0
69. —— £3 - 11 - 0	71	0	0
70. —— £1 - 4 - 9	24	15	0
71. —— 7*s.* 4*d.*	7	6	8
72. 60 at 13*s.* 0*d.*	39	0	0
73. 180 at £1 - 11 - 0	279	0	0
74. 82 at 16*s.* 6*d.*	67	13	0

EXERCISES ON RULE IV.

Exercises.	*Answers.* £	s.	d.
75. 112 at 4*d.*	1	17	4
76. —— 7*d.*	3	5	4
77. —— 1*s.* 8*d.*	9	6	8
78. —— 9½*d.*	4	8	8
79. —— 10¾*d.*	5	0	4

EXERCISES ON RULE V.

Exercises.	*Answers.* £	s.	d.
80. 120 at $5d.$	2	10	0
81. —— $8d.$	4	0	0
82. —— $4s.\ 6d.$	27	0	0
83. —— $10\frac{1}{4}d.$	5	2	6
84. —— $1s.\ 2\frac{1}{2}d.$	7	5	0
85. —— $1s.\ 7\frac{3}{4}d.$	9	17	6

EXERCISES ON RULE VI.

Exercises.	*Answers.* £	s.	d.
86. 100 at $7s.\ 0d.$	35	0	0
87. —— $12s.\ 6d.$	62	10	0
88. —— $5s.\ 9d.$	28	15	0
89. —— $11s.\ 4d.$	56	13	4

EXERCISES ON RULE VII.

Exercises.	*Answers.* £	s.	d.
90. 100 at $2\frac{1}{2}d.$	1	0	10
91. —— $5\frac{1}{4}d.$	2	3	9
92. —— $3\frac{3}{4}d.$	1	11	3
93. —— $9\frac{1}{4}d.$	3	17	1
94. —— $11\frac{1}{4}d.$	4	13	9

EXERCISES ON RULE VIII.

Exercises.	*Answers.* £	s.	d.
95. 112 at $2\frac{3}{4}d.$	1	5	8
96. 1000 at $1\frac{1}{4}d.$	5	4	2
97. 365 at $2\frac{1}{4}d.$	3	8	$5\frac{1}{4}$
98. 960 at $2s.\ 8\frac{1}{2}d.$	130	0	0
99. 148 at $7d.$	4	6	4
100. 200 at $10d.$	8	6	8
101. 253 at $1s.\ 1d.$	13	14	1
102. 144 at $2s.\ 1d.$	15	0	0

Exercises.	*Answers.* £	s.	d.
103. 150 at 7*s.*	52	10	0
104. 440 at 11*s.*	242	0	0
105. 241 at 13*s.*	156	13	0
106. 380 at £2 - 15	1045	0	0
107. 352½ at 3*s.*	52	17	6
108. 365 at 1¾*d.*	2	13	2¾
109. —— 7*d.*	10	12	11
110. —— 7*s.*	127	15	0
111. 313 at 2¾*d.*	3	11	8¾
112. —— 11*d.*	14	6	11
113. —— 11*s.*	172	3	0

EXERCISES ON RULE IX.

Exercises.	*Answers.* £	s.	d.
114. £24 for 8 mos., at 5 per c. per ann.	0	16	0
115. £55 for 5 ——————	1	2	11
116. £84 for 11 ——————	3	17	0
117. £132 for 4 ——————	2	4	0
118. £90 for 2 ——————	0	15	0
119. £75 for 9 ——————	2	16	3
120. £112 for 10 ——————	4	13	4
121. £33 for 13 ——————	1	15	9
122. £60 for 7 mos., at 6 per c. per ann.	2	2	0
123. £66 for 5 mos., at 4 ——————	1	2	0
124. £125 for 9 mos., at 4½ ——————	4	4	4½
125. £240 for 11 mos., at 3 ——————	6	12	0

EXERCISES ON RULE X.

Exercises.	*Answers.* £	s.	d.
126. £16 for 3 mos., at 6 per c. per ann.	0	4	9½
127. £45 for 5 ——————	1	2	6
128. £88 for 7 ——————	3	1	7¼
129. £133 for 11 ——————	7	6	3¹
130. £65 for 7 mos., at 4 per c. per ann.	1	10	4
131. £52 - 10 for 10 mos., at 4½ ——————	1	19	4½

EXERCISES ON RULE XI.

	Exercises.	*Answers.* £	s.	d.
132.	£18 for 120 days, at 5 per c. per ann.	0	5	11
133.	£64 for 60 days, at 5 ———	0	10	$6\frac{1}{4}$
134.	£67 - 10 for 24 days, at 5 ———	0	4	$5\frac{1}{4}$
135.	£160 for 25 days, at 5———	0	10	$11\frac{1}{2}$
136.	£224 for 60 days, at 5———	1	16	$9\frac{3}{4}$
137.	£150 for 150 days, at 5———	3	1	$7\frac{3}{4}$
138.	£84 for 40 days, at 6 ———	0	11	$0\frac{1}{2}$
139.	£270 for 114 days, at 4———	3	7	$5\frac{1}{2}$
140.	£325 for 150 days, at $5\frac{1}{2}$ ———	7	6	11

MISCELLANEOUS EXERCISES.

Exer. 141. Find $\frac{3}{16}$ of 112 and $\frac{5}{17}$ of 289: find also the sum and difference of the results. *Answ.* 21, 85, 106, and 64.

142. What is the number of which 42 is $\frac{3}{11}$? *Answ.* 154.

143. Find $\frac{7}{12}$ of 576 and $\frac{17}{24}$ of the result. *Answ.* 336 and 238.

144. Square 38 and cube 16. *Answ.* 1444 and 4096.

145. Find a fourth proportional to 28, 42, and 128. *Answ.* 192.

146. Required the prices of 64 lbs. of butter, at $10\frac{1}{4}$*d.* per lb.; and of 31 lbs. at $9\frac{1}{2}$*d.* *Answ.* £2 - 14 - 8, and £1 - 4 - $6\frac{1}{2}$.

147. Find the prices of 47 barrels of oats at 10*s.* 6*d.* per barrel, and $38\frac{1}{2}$ barrels of barley at 12*s.* 9*d.* per barrel. *Answ.* £24 - 13 - 6, and £24 - 10 - $10\frac{1}{2}$.

148. What cost 49 lbs. of sugar at £3 - 16 per cwt., and 48 lbs. at £3 - 17 per cwt.? *Answ.* £1 - 13 - 3, and £1 - 13 - 0.

149. What are the costs of $15\frac{1}{2}$ lbs. of beef, and $13\frac{1}{4}$ lbs. of mutton, the first at 7*d.* and the second at 6*d.* per lb.? *Answ.* 9*s.* $0\frac{1}{2}$*d.* and 6*s.* $7\frac{1}{2}$*d.*

150. Find the price of $9\frac{3}{4}$ lbs. of salmon at 1*s.* 6*d.* per lb. *Answ.* 14*s.* $7\frac{1}{2}$*d.*

151. Reduce £99 - 19 - 2 to pence. *Answ.* 23990.

152. Reduce 1245 pence to pounds. *Answ.* £5 - 3 - 9.

153. If $\frac{3}{8}$ of a gallon of wine cost 7*s.* 6*d.*, what cost 1 gallon and 89 gallons? *Answ.* £1 and £89.

154. If $\frac{5}{8}$ of a ton cost £3 - 15, what cost 1 ton, and $12\frac{3}{4}$ tons? *Answ.* £6, and £76 - 10 - 0.

155. If a hundred weight of sugar cost £2 - 10 - 0, what is gained by selling it at 7*d.* per lb.? *Answ.* 15*s.* 4*d.*

156. What is gained or lost by selling 122 yards of broadcloth at 18*s.* per yard, the first cost and charges having amounted to £110? *Answ.* Lost 4*s.*

157. If a person drink, daily, a bottle of porter, worth $4\frac{1}{2}$*d.* what is the amount for a year? *Answ.* £6 - 16 - $10\frac{1}{2}$.

158. If 12 bottles of wine cost £1 - 7, what cost 7 dozen and 5 bottles? *Answ.* £10 - 0 - 3.

159. Reduce 742 guineas to pounds. *Answ.* £779 - 2 - 0.

160. If one steamer start at 12 o'clock, and sail at the rate of $10\frac{1}{2}$ miles per hour, and another start at half past one, and sail in the same direction, at the rate of $11\frac{1}{4}$ miles per hour; at what hour will the latter overtake the former? *Answ.* At half past ten on the succeeding day.

161. How many times greater is a square field, having each side 60 perches, than a triangular one having its base = 18 perches, and its perpendicular = 10 perches?* *Answ.* 40 times.

162. What is the interest of £65 for 1 year, 5 months, at 5 per cent. per annum? *Answ.* £4 - 12 - 1.

163. Find the present worth of a bill for £95, due at the end of $4\frac{1}{2}$ months, interest at 4 per cent. being taken as discount. *Answ.* £93 - 11 - 6.

164. A person borrows £100, and pays £55 at the end of 7 months, and the rest at the end of a year. What has he to pay for interest at 5 per cent. per annum? *Answ.* £3 - 17 - 1.

165. Find the interest of £70 from the 4th of April till the 24th of July, at 6 per cent. per annum. *Answ.* £1 - 5 - $6\frac{1}{2}$.

166. Find the present worth at $4\frac{1}{2}$ per cent. per annum, of a bill for £43 - 6 - 8, which has 60 days to run. *Answ.* £43 - 0 - 5.

167. Required the present worth of £250, due at the end of 160 days, at $5\frac{1}{2}$ per cent. per annum. *Answ.* £243 - 19 - $5\frac{1}{2}$.

168. How often will a carriage wheel, $13\frac{3}{4}$ feet in circumference, revolve in going a mile? *Answ.* 384 times.

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* This question will, of course, be omitted by those who are unacquainted with mensuration.

THE METRIC SYSTEM OF WEIGHTS AND MEASURES.

The tables of weights and measures already given, afford a remarkable exemplification of the utter absurdity, and great, and wasteful, and dangerous inconvenience, of the variety of systems, in more or less common use in Great Britain, for the reckoning of mass and dimensions. Thus different modes of reckoning mass are employed in commerce and the arts, according as the substance is one of the precious metals, or an article of ordinary merchandise, or a medicine specified in a physician's prescription. The apothecary buys according to avoirdupois weight,—pounds of 16 of one kind of ounce ($437\frac{1}{2}$ grains), and drams, and grains; and compounds his medicines according to apothecaries' weight,—pounds of 12 of another kind of ounce (480 grains), and drams, and scruples, and grains. The avoirdupois dram is $27\frac{11}{32}$ grains; the apothecaries' dram is 60 grains. A certain multiple of the imperial unit of length, called a mile, is used for the measurement of distance travelled by land and for land telegraph wires; while another unit of length, also called a mile, which is one minute of latitude, or one minute of longitude at the equator, on the earth's surface, and is not an exact multiple of the imperial unit of length, is used for the measurement of distance travelled by sea and for the measurement of submarine telegraph wires. The length of a rope or chain is measured in yards, feet, or inches, the length of a piece of cloth in yards, quarters, or nails. These various modes of reckoning, since they have no connection with one another, and follow no method in the formation of multiples and submultiples of their fundamental units, are a source of great inconvenience to the British nation, which

might be wholly avoided by the adoption of some simple and uniform mode of reckoning with different denominations chosen according to the decimal system. Such a mode of reckoning has been long in use for general purposes in France, Germany, and Italy, and for scientific purposes throughout the whole world. It owes its origin to a decree * of the French Republic passed in 1795, which, giving effect to a recommendation of a committee of the French Academy of Sciences, provided that the unit of length should be one ten-millionth part of the quadrant of the earth's circumference measured along the meridian of Paris. The work of realising this standard was entrusted to Borda, who, using the determination of the length of an arc of the meridian made by Delambre and Méchain, constructed a rod of platinum which, when at a temperature of 32° Fahrenheit, or 0° Centigrade, represented, as accurately as geodetical knowledge then permitted, the length specified by the decree. This rod has been carefully preserved in the national archives of France, and by it is defined the metre or fundamental unit of the French system now called the metric system. The length of the metre, it is to be observed, therefore, is not affected by the results of any subsequent geodesy, though more accurate than that of Delambre and Méchain; on the contrary, all such results are expressed in terms of the length of Borda's platinum rod.

One great convenience of the French metric system consists in the fact that the multiples and submultiples of its fundamental unit follow the decimal law. Thus the metre, which is equal in length to 39·37079 English inches, is divided into 10 parts, each of which is called a centimetre, and each centimetre into 10 parts, each of which is called a millimetre; again, a length of 10 metres is called a decametre, a length of 100 metres a hectometre, a length of 1000 metres a kilometre, and so on. Thus any length or distance whatever can be expressed in convenient numbers; and at the same time the utmost facility of computation is obtained by the adoption of the decimal scale of numeration.

Another advantage of the French metric system is

* Loi du 18 germinal, an III.

found in the relation of the unit of length to the unit of mass or weight adopted in that system. The French unit of mass or weight is defined as the mass of a certain piece of platinum, called the "Kilogramme des Archives," which was also made by Borda, and was intended to represent the mass of a cubic decimetre of distilled water at the temperature of maximum density, viz. 39° Fahrenheit or 4° Centigrade. Although, when the utmost strictness of definition is required, it must be remembered that the standard of mass is the mass of that one piece of platinum, in most cases no sensible error is introduced by estimating the mass of a cubic decimetre of water at 39° Fahrenheit, or at 4° Centigrade, as a kilogramme, or the mass of a cubic centimetre of water at that temperature as one gramme. A great simplification is by this means introduced into the computation of masses of different volumes of various substances from their specific gravities.

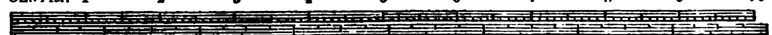
The specific gravity of a substance is the ratio of the mass of a certain volume of the substance to the mass of an equal volume of water at its temperature of maximum density. Hence if the specific gravity of a substance be known, the mass of any specified volume of the substance is found by multiplying the number of units of volume by the specific gravity, and the result by the mass of unit volume of water. This last factor, which, in the English system, is always a troublesome one, is reduced to unity in the metric system by the connection explained above between the unit of length and the unit of mass. Thus the mass of a given volume of a substance is given in grammes by the product of the specific gravity and the volume in cubic centimetres, in kilogrammes by the product of the specific gravity and the volume in cubic decimetres or litres, and in French tonnes by the product of the specific gravity and the volume in cubic metres. The French tonne, or 1000 kilogrammes, is ·984 of the English ton.

The following tables show the relations of the various weights and measures of the metric system to one another and their values in English imperial units.

FRENCH MEASURES OF LENGTH.

	In English inches	In English feet =12 inches	In English yards =3 feet	In English fathoms =6 feet	In English miles =1760 yards
Millimetre	0·03937	0·003281	0·0010936	0·0005468	0·0000006
Centimetre	0·39371	0·032809	0·0109363	0·0054682	0·0000062
Decimetre	3·93708	0·328090	0·1093633	0·0546816	0·0000621
Metre	39·37079	3·280899	1·0936331	0·5468165	0·0006214
Decametre	393·70790	32·808992	10·9363306	5·4681653	0·0062138
Hectometre	3937·07900	328·089917	109·3633056	54·6816528	0·0621382
Kilometre	39370·79000	3280·899167	1093·6330556	546·8165278	0·6213824
Myriametre	393707·90000	32808·991667	10936·3305556	5468·1652778	6·2138242

1 inch=2·539954 centimetres. 1 foot=3·0479449 decimetres. 1 yard=0·9143835 metre. 1 mile=1·6093149 kilometre.



FRENCH MEASURES OF SURFACE.

	In English square feet	In English sq. yds. =9 square feet	In English poles =272·25 sq. feet	In English roods =10,890 sq. feet	In English acres =43,560 sq. feet
Centiare, or square metre	10·764299	1·196033	0·0395383	0·0009885	0·0002471
Are, or 100 square metres	1076·429934	119·603326	3·9538290	0·0988457	0·0247114
Hectare, or 10,000 sq. metres	107642·993419	11960·332602	395·3828959	9·8845724	2·4711431

1 square inch=6·4513669 square centimetres. 1 square yard=0·83609715 square metre or centiare.
1 square foot=9·2899683 square decimetres. 1 acre =0·40467102 hectare.
1 square mile=2·58989451 square kilometres.

FRENCH MEASURES OF CAPACITY.

	In cubic inches	In cubic feet =1728 cub. in.	In pts.=34·65923 cubic inches	In gallons=8 pts. =277·27384 cub. in.	In bush.=8 gals.= 2218·19075 cub. in.
Millilitre, or cubic centimetre	0·06103	0·000035	0·00176	0·0002201	0·0000275
Centilitre, or 10 cubic centimetres	0·61027	0·000353	0·01761	0·0022010	0·0002751
Decilitre, or 100 cubic centimetres	6·10271	0·003532	0·17608	0·0220097	0·0027512
Litre, or cubic decimetre	61·02705	0·035317	1·76077	0·2200967	0·0275121
Decalitre, or centistere	610·27052	0·353166	17·60773	2·2009668	0·2751208
Hectolitre, or decistere	6102·70515	3·531658	176·07734	22·0096677	2·7512085
Kilolitre, or stere, or cubic metre	61027·05152	35·316581	1760·77341	220·0966767	27·5120846
Myrialitre, or decastere	610270·51519	353·165807	17607·73414	2200·9667675	275·1208459

1 cubic inch=16·386176 cubic centimetres. 1 cubic foot=28·315312 cubic decimetres. 1 gallon=4·543458 litres.

FRENCH MEASURES OF WEIGHT OR MASS.

	In English grains	In troy ounces =480 grains	In avoirdupois lbs. =7000 grains	In cwts.=112 lbs. =784,000 grains	Tons=20 cwts. = 15,680,000 grains
Milligramme	0·01543	0·000032	0·0000022	0·0000000	0·0000000
Centigramme	0·15432	0·000322	0·0000220	0·0000002	0·0000000
Decigramme	1·54323	0·003215	0·0002205	0·0000020	0·0000001
Gramme	15·43235	0·032151	0·0022046	0·0000197	0·0000010
Decagramme	154·32349	0·321507	0·0220462	0·0001968	0·0000098
Hectogramme	1543·23488	3·215073	0·2204621	0·0019684	0·0000984
Kilogramme	15432·34880	32·150727	2·2046213	0·0196841	0·0009842
Myriagramme	154323·48800	321·507267	22·0462126	0·1968412	0·0098421

1 grain=0·064799 gramme. 1 troy oz.=31·103496 gram. 1 lb. avoird.=0 453593 kilogr. 1 cwt.=50·802377 kilogrs.

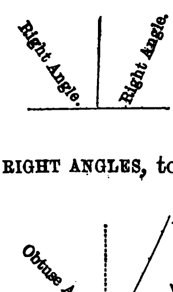
APPENDIX,

CONTAINING

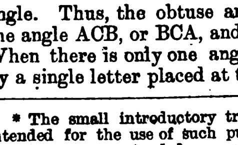
AN INTRODUCTION TO MENSURATION.*

DEFINITIONS.

I. An Angle is the mutual inclination of two straight lines that meet one another in a point, which is called the vertex of the angle: or it is the degree of their opening or divergence.



II. When one straight line standing on another, makes with it two angles which are equal, each of these angles is called a right angle; and the straight line which stands on the other is said to be perpendicular, or at right angles, to it, or to be a perpendicular to it.



III. An obtuse angle is greater than a right angle: an acute angle is less than a right angle.

An angle is usually named by three letters, the middle one being placed at the vertex, and the other two somewhere on the lines which contain the angle. Thus, the obtuse angle in the last diagram, may be called the angle ACB, or BCA, and the acute one the angle ACD, or DCA. When there is only one angle at the same point, it is best named by a single letter placed at that point.

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* The small introductory tract on mensuration which is here given, is intended for the use of such pupils as may not have time or opportunity for studying a more extended course. For this reason, the easiest and most useful parts are selected, and the definitions and illustrations are delivered in plain and familiar terms, rather than with a view to mathematical precision. The rules are also given without demonstrations, as the pupils for whom this abstract is intended are not supposed to have read a preparatory course of mathematics. Those who may wish to prosecute the subject more extensively, may have recourse to any good modern treatise on the subject.

IV. PARALLEL STRAIGHT LINES are those which have everywhere equal perpendicular distances from each other.

V. A SURFACE, or SUPERFICIES, has length and breadth without thickness.

VI. A BODY, or, as it is often called, a SOLID, has length, breadth, and thickness, or depth.

VII. A FIGURE is a portion of space inclosed by one or more boundaries.

VIII. A FIGURE is said to be EQUILATERAL, if it have equal sides; and EQUIANGULAR, if it have equal angles.

IX. If a figure be contained by three lines, it is called a TRIANGLE; if by four, a QUADRILATERAL; if by more than four, a POLYGON.

X. An equilateral and equiangular polygon is often called a REGULAR POLYGON.

XI. Polygons of five, six, seven, eight, nine, ten, eleven, and twelve sides, are often called, respectively, PENTAGONS, HEXAGONS, HEPTAGONS, OCTAGONS, ENNEAGONS, or NONAGONS, DECAGONS, HENDECAGONS, and DODECAGONS.

XII. A quadrilateral which has its opposite sides parallel, is called a PARALLELOGRAM: and a parallelogram which has its angles right angles, is called a RECTANGLE.

XIII. A quadrilateral which has its sides equal, and its angles right angles, is termed a SQUARE; and a quadrilateral which has its sides equal, but its angles not right angles, is called a RHOMBUS.

XIV. A quadrilateral which has two sides parallel, and the other two not, is called a TRAPEZOID.

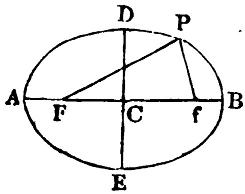
XV. Any quadrilateral, except a parallelogram or trapezoid, is termed a TRAPEZIUM.

XVI. A DIAGONAL of a figure is a straight line passing through two of its angles which are not adjacent to one another.

XVII. A CIRCLE is a figure contained, on a flat surface, by one line which is called the CIRCUMFERENCE; and is such that all straight lines drawn to the circumference from a certain point within the figure, called the CENTRE, are equal to each other. Any of those equal lines is called a RADIUS: and a line drawn through the centre, and terminated both ways by the circumference,

is called a DIAMETER. Hence, a diameter is evidently double of a radius.

XVIII. If the ends of a thread, FPf, be fastened to two pins, F, f, fixed at a less distance asunder than the length of the thread, and if the point of a pen or pencil, P, be carried round in such a manner as to keep the thread constantly stretched, the curve line thus described, is called an ELLIPSE; the points F and f, where the pins are fixed, are called the FOCI (and each of them a FOCUS); the line AB, drawn through the foci, and terminated both ways by the curve, is called the GREATER AXIS; and the line DE, drawn at right angles to this axis through its middle point, and terminated by the curve, is called the LESS AXIS.



XIX. MENSURATION is the method of determining by computation the comparative magnitudes of figures; and it is divided into two great branches, *the mensuration of surfaces*, and *the mensuration of bodies or solids.*

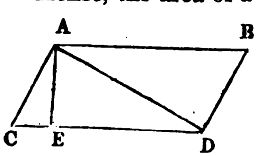
XX. The AREA of a surface is the space which it contains. In mensuration, the magnitude of this space is ascertained by the number of times that a given space, called the *measuring unit*, is contained in it.

XXI. The MEASURING UNIT which is adopted for surfaces, is a square whose side is some of the common measures of length, such as *a square inch*, *a square foot*, *a square yard*, &c.—(See the table of square measure, page 59.)

MENSURATION OF SURFACES.

RULE I. *To find the area of a parallelogram:* Multiply its length by its perpendicular breadth.

Hence, the area of a square is found simply by multiplying a side by itself.



Exam. 1. Required the area of the parallelogram ABDC, the length AB, or CD, being 36 feet, and the perpendicular breadth, AE, 15 feet.

Here the product of 36 and 15 is 540, the area in square feet.

Exam. 2. If the length of the floor of a rectangular room be 22 feet 5 inches, and its breadth 16 feet 11 inches, how many square feet does it contain?

This and similar examples, in which the dimensions are given in feet and inches, may be wrought in different ways.

1. By reducing 5 inches, and 11 inches, each to the decimal of a foot, we obtain for the dimensions, $22{\cdot}41\dot{6}$, and $16{\cdot}91\dot{6}$; the product of which is $379{\cdot}2152\dot{7}$ square feet, or $379\frac{1}{5}$ square feet nearly, the area required.

2. As a second method, by reducing the dimensions to inches, we obtain 269 and 203; the product of which is 54607 square inches, the area required: and by dividing this by 144, we get 379 square feet, 31 square inches, a result which agrees with that found by the former method.

3. Another mode, which is perhaps preferable to either of the preceding, is the method of *duodecimals*, or, as it is often improperly called, the method of *cross multiplication*. This will be understood from the operation in the margin, in which we multiply successively by 16 and 11. The product of 16 and 5 is divided by 12, and the remainder is set down, and the quotient carried. In multiplying by 11, both the products are divided by 12, and the remainders set one place towards the right hand. The sum of the two partial products is then taken, and the result is found to be 379 square feet, 2 twelfths of a square foot, or 24 square inches, and 7 square inches; or 379 square feet, 31 square inches, as before. The answer might also be obtained by multiplying by 16, and working for 11 inches by means of aliquot parts.

Feet	In.	
22	5	
16	11	
358	8	
20	6	7
379	2	7

The results obtained by this latter method are often called, very improperly, *feet, inches*, and *parts*, instead of square feet, twelfths of such feet, and square inches. According to this mode, the last answer would be read 379 feet, 2 inches, and 7 parts. It may be written, 379 f. 2′ 7″.

The multiplication of feet and inches might also be performed very simply and easily, by reducing the feet to the duodecimal scale and then carrying for 12 in the multiplication. Thus, by reduction to the duodecimal scale (see page 298), the dimensions in the foregoing example become 1D·5 and 14·H, and the work will stand as in the margin, the answer in the duodecimal scale being 277·27, or by reduction to the decimal scale, 379 feet, 2 twelfths of a square foot, and 7 inches. The learner, after a little practice, would probably prefer this method to any other.

1D·5
14·H
1867
758
1D5
277·27, or
379 f. 2′ 7″ *answ.*

Exam. 3. Required the area of a square field, each of whose sides is 6 chains 30 links.

Here (note 3, page 59), each side is 630 links; and multiplying this by itself, we obtain for the area 396900 square links; or 3·969 acres, by division by 100,000: whence, by multiplying by 4 and 40, we get for the required content 3 acres, 3 roods, 35 perches, with a small remainder.

Links
630
630
189
378
3·96900
4
3·876
40
35·040

Exer. 1. Required the content of a field in form of a parallelogram, the length and breadth of which are 12 chains 76 links, and 9 chains 43 links. *Answ.* 12 acres, 0 roods, 5 perches.

2. Given the length of a street = 937 feet 6 inches, and its breadth = 66 feet 8 inches; required the cost of paving it, at $8\frac{1}{2}$*d.* per square yard. *Answ.* £245 - 18 - $11\frac{3}{4}$.

3. At $9\frac{3}{4}$*d.* per yard, required the cost of painting the walls of a room, the sum of the lengths of whose sides is 70 feet 10 inches, and its height 10 feet 1 inch. *Answ.* £3 - 4 - $5\frac{3}{4}$.

4. Required the content of a rectangular garden whose length is 98 yards, and breadth 81 yards. *Answ.* 1 acre, 2 roods, 22 perches, $12\frac{1}{2}$ yards.

5. What is the content of a deal-board, 9 feet 8 inches long, and $8\frac{1}{2}$ inches broad? *Answ.* 6 f. 10′ 2″.

6. If each side of a square table be 3 feet 10 inches, what is its content? *Answ.* 14 f. 8′ 4″.

RULE II. *To find the area of a triangle:* Multiply the base by the perpendicular, and take half the product: or multiply one of these lines by half the other.

Exam. 4. Required the area of the triangle ACD, whose base CD is 15 feet 4 inches, and perpendicular AE, 8 feet 7 inches.

Here, by the method of duodecimals already explained and by halving, we find for the area, 65 feet, 9 twelftns of a foot, and 8 inches, or $65\frac{3}{4}$ feet, nearly.

	Feet	In.	
	15	4	
	8	7	
	122	8	
	8	11	4
2)	131	7	4
	65	9	8

Exer. 7. Given the base of a triangle = 13 chains 24 links, and its perpendicular = 8 chains 59 links; to find its area. *Answ.* 5 a. 2 r. 30 p.

8. If the base of a triangle be 21 feet 7 inches, and its perpendicular 17 feet 10 inches, what is its area? *Answ.* 192 f. 5′ 5″.

RULE III. *To find the area of a triangle, when the three sides are given:* (1.) Add the sides tog

take half the sum: (2.) From the half sum take the three sides severally: (3.) Find the continual product of the half sum and the three remainders: (4.) Extract the square root of this product.

The area of an equilateral triangle may be found by multiplying the square of one of the sides by ·4330127.

Exam. 5. Required the area of a triangle, whose sides are 2, 3, and 4 feet, respectively.

Here, half the sum of the sides is 4·5; and subtracting from this the three sides successively, we find the three remainders 2·5, 1·5, and ·5. Then $4{\cdot}5 \times 2{\cdot}5 \times 1{\cdot}5 \times {\cdot}5 = 8{\cdot}4375$; and $\sqrt{8{\cdot}4375} = 2{\cdot}9047375$, the area in square feet.

Exer. 9. Given the sides of a triangle equal to 9 chains 62 links, 6 chains 38 links, and 7 chains 20 links; to find its area. *Answ.* 2 acres, 1 rood, $7\frac{1}{2}$ perches, nearly.

10. If the sides of a triangle be 13, 14, and 15 feet, respectively, what is its area? *Answ.* 84 feet, or $9\frac{1}{3}$ yards.

11. Given the sides of a triangle equal to 3 feet 8 inches, 4 feet 7 inches, and 6 feet 5 inches, respectively; to find the area. *Answ.* 8·233 feet.

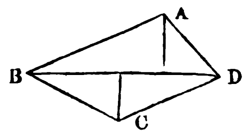
Rule IV. *To find the area of a trapezoid:* Multiply the sum of the parallel sides by the perpendicular breadth of the figure, and take half the product.

Exer. 12. Given the parallel sides of a trapezoid = 33 and 28 inches, and its breadth = 11 inches; required the area. *Answ.* $335\frac{1}{2}$ inches.

13. If the parallel sides of a trapezoid be 75 and 33 feet, and its breadth 20 feet, what is its area? *Answ.* 1080 feet, or 120 yards.

Rule V. *To find the area of a trapezium:* Multiply either of the diagonals by the sum of the perpendiculars drawn to it from the opposite angles, and take half the product. Or,

Find, by Rule II. or III., the areas of the triangles that compose the trapezium, and add them together.



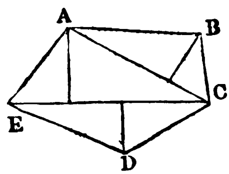
Exam. 6. Given BD, the diagonal of the trapezium ABCD, = 16 perches; and the perpendiculars drawn from A and C = 7 perches, and 5 perches; to find the area.

m of the perpendiculars is 12 perches; and the dia- ›erches, we have $12 \times 16 = 192$, the half of which is ›quare perches.

Exer. 14. Given one of the diagonals of a trapezium = 17 chains, 56 links, to compute the area; the perpendiculars to that diagonal from the opposite angles being 8 chains, 82 links, and 7 chains, 73 links. *Answ.* 14 acres, 2 roods, 5 perches.

15. Given BD = 21 perches, BA = 18 perches, AD = 9 perches, BC = 12 perches, and CD = 15 perches; to find the area. *Answ.* 168·68008 perches.

RULE VI. *To find the area of a polygon:* (1.) Divide it by diagonals into triangles and trapeziums. (2.) Find the areas of these by some of the foregoing rules. (3.) Add all these areas together.



Exer. 16. Given EC = 58 perches, and the perpendiculars to it from A and D = 27 and 18 perches respectively; given also AC = 46 perches, and the perpendicular to it from B = 20 perches; to find the area. *Answ.* 11 a. 0 r. 5 p.

17. Given AB = 6 feet, BC = 3 feet, CD = 4 feet, DE = 5 feet, AE = 4 feet, AC = 7 feet, and EC = 8 feet; required the area. *Answ.* 31·123574 feet.

RULE VII. The *area of a regular polygon* is most easily found by multiplying the square of one of the sides by the number standing opposite to the name of the polygon in the annexed table.

Pentagon............	1·7204774	Enneagon............	6·1818240
Hexagon............	2·5980762	Decagon	7·6942088
Heptagon	3·6339126	Hendecagon	9·3656411
Octagon	4·8284272	Dodecagon	11·1961524

Exer. 18. To find the area of a regular hexagon, having each of its sides = 5 inches. *Answ.* 64·951905 square inches.

19. Given the side of a regular octagonal inclosure = 60 yards; to compute the area. *Answ.* 3 a. 2 r. 14 p. 19 yds.

RULE VIII. *The diameter of a circle being given, to find the circumference:* Multiply the diameter by 3·1416; or, for greater accuracy, by 3·141593: Or,

As 113 is to 355, so is the diameter to the circumference: or, when much accuracy is not necessary, as 7 is to 22, so is the diameter to the circumference.

The diameter would be found from the circumference, by reversing any of the foregoing processes.

Exer. 20. Given the diameter of a circle = 13 inches; to find its circumference. *Answ.* 40·8407.

21. If the diameter of a circular cask be 2 feet 9 inches, what will be the length of a hoop for it? *Answ.* 8 feet 7·67257 inches.

22. If the girt of a round tree be 12 feet 5 inches, what is its diameter? *Answ.* 3 feet 11·42817 inches.

RULE IX. *To find the area of a circle:* Multiply the diameter by the circumference, and take one fourth of the product: Or,

Multiply the square of the diameter by 0·7854; or, for greater accuracy, by 0·785398: Or,

Multiply the square of the radius by 3·141593: Or,

Multiply the square of the circumference by 0·07958.

The *area of an ellipse* is found by multiplying the product of the axes by 0·7854, or more accurately, 0·785398.

Exer. 23. Required the area of a circular pond whose diameter is 31 yards. *Answ.* 754·7694 square yards.

24. Required the area of the space on which a horse may graze, when confined by a cord $7\frac{1}{2}$ perches in length, having one of its ends fixed at a certain point. *Answ.* 1 a. 0 r. 16·7 p.

25. Required the content of a circular grove, 56·5 perches in circumference. *Answ.* 1 a. 2 r. 14 p.

26. What is the area of a circular table whose diameter is 5 feet 8 inches? *Answ.* 25 feet $31\frac{2}{3}$ inches nearly.

27. What is the area of an elliptic ceiling, the axes of which are 33 feet 5 inches, and 20 feet 3 inches? *Answ.* 59 yards, 0 feet, 67 inches.

MENSURATION OF BODIES, OR SOLIDS.

DEFINITIONS.

I. A figure whose ends, or bases, are parallel, and whose sides are parallelograms, is called a PRISM. Such a figure is termed a RIGHT PRISM, if each of its bases be perpendicular to its other sides; and it is said to be TRIANGULAR, HEXAGONAL, &c., if its bases be triangles, hexagons, &c.

II. A prism whose bases, as well as its other sides, are parallelograms, is called a PARALLELEPIPED. A common chest, a bar of iron, a brick, &c., afford instances of *right parallelepipeds.*

III. A right parallelepiped whose sides are equal squares, is called a CUBE. Such are dice.

IV. A PYRAMID is a body bounded by plane surfaces, meeting in a point called the VERTEX of the pyramid, and by a rectilineal base terminated by those planes.

V. A CYLINDER is a round body which is of equal thickness throughout, and which has circular bases, parallel to each other, such as a rolling stone for walks, a round pillar, &c.

VI. A body which has a circular base, and which tapers uniformly to a point named the VERTEX, is called a CONE. A sugar-loaf, and the tapering top of a round pillar are nearly of this form.

VII. A FRUSTUM of a body is what remains, when the top is cut off by a plane parallel to the base.

VIII. A GLOBE, or SPHERE, is a body of such a figure that all points of the surface are equally distant from a point within it called the centre.

IX. If one of the parts into which an ellipse is divided by either of the axes, revolve about that axis, the figure which it describes is called a SPHEROID:—PROLATE, if the revolution be performed round the greater axis; OBLATE, if round the less. An egg is nearly of the former figure; and a watch, or a flat turnip, nearly of the latter.

X. The CONTENT, or VOLUME, or, as it is often improperly called, the SOLIDITY of a body, is the space contained within it. The magnitude of this space is expressed by the number of times that it contains a given space, called the *measuring unit.*

XI. The MEASURING UNIT which is adopted for bodies, is a cube whose base is the measuring unit for surfaces, such as a cubic inch, a cubic foot, &c.—(See the table of cubic measure, page 60.)

RULE I. *To find the content of a prism or cylinder:* (1.) Find the area of the base: (2.) Multiply this by the perpendicular height, or distance between the ends.

Hence, *to find the content of a right parallelepiped,* take the continual product of the length, breadth, and thickness, or depth: and, *to find the content of a cube,* find the third power of one of the sides of its base.

Exam. 1. Required the content of a box of cloth, the length, breadth, and depth of which are 4 feet 10 inches, 2 feet 11 inches, and 2 feet 2 inches, respectively.

Here, by the method of duodecimals, we multiply 4 feet 10 inches by 2 feet 11 inches: the product, 14 feet, 1 twelfth, and 2 inches, is the area of the base. We then multiply in a similar manner by 2 feet 2 inches, the depth, setting each product in multiplying by the inches, one place towards the right hand. The final product, or content of the box, is 30 cubic feet, 6 twelfths of a foot, 6 one-hundred-and-forty-fourths of a foot, and 4 inches, or 30½ feet, nearly. In measures of capacity, since the cubic foot contains 1728 cubic inches, the twelfths of a foot are each 144 cubic inches, and the hundred-and-forty-fourths are each 12 inches. Hence the foregoing answer is easily reduced to 30 feet and 940 inches. The same result would be obtained by reducing the dimensions to inches, finding their continual product, and dividing it by 1728; or by reducing the inches to decimals, and proceeding in a similar manner. The results in such cases as the present, are often improperly called feet, inches, parts, and seconds.

Feet	In.		
4	10		
2	11		
9	8		
4	5	2	
14	1	2	
2	2		
28	2	4	
2	4	2	4
30	6′	6″	4‴

Exer. 1. Required the number of gallons of water, of 277·274 cubic inches each, contained in a rectangular cistern, the length, breadth, and depth of which are 16 feet, 10 feet 6 inches, and 8 feet 4 inches. *Answ.* 8724·94.

2. If each side of the base of a triangular prism be 2 inches, and its length 14 inches, what is its content? *Answ.* 24·2487 inches.

3. Required the number of cubic feet contained in a room whose length, breadth, and height are 24 feet, 18 feet 6 inches, and 10 feet 7 inches. *Answ.* 4699.

4. Given the diameter of the base of a cylindric column = 3 feet 1 inch, and its height = 18 feet 9 inches; to find its content. *Answ.* 140·0016 feet.

5. Required the content of a bale, the length, breadth, and thickness of which are 4 feet 8 inches, 3 feet 3 inches, and 2 feet 6 inches. *Answ.* 37 feet, 11 twelfths.

6. Given the length, breadth, and thickness of a uniform plank equal to 22 feet 7 inches, 1 foot 5 inches, and 6½ inches respectively; to find the content. *Answ.* 17·32957 feet.

RULE II. *To find the content of a pyramid or cone.* Multiply the area of the base by the perpendicular height, and take one third of the product.

Exer. 7. Given each side of the base of a square pyramid = 10 inches, and the perpendicular height, or altitude = 9 feet 9 inches; required the content. *Answ.* 2 f. 3′ 1″.

8. Given the diameter of the base of a conical glass-house $=37$ feet 8 inches, and its height $=79$ feet 9 inches; required the entire space inclosed. *Answ.* 29622 feet, nearly.

9. The height of the largest of the Egyptian pyramids is 477 feet, and each side of its base, which is a square, is 720 feet; required the content. *Answ.* 82425600 cubic feet, or 3052800 cubic yards.

10. Given the height of a conical sugar-loaf $=17$ inches, and the diameter of its base $=9$ inches; required the content. *Answ.* 360·4986 inches.

11. How often may a conical glass, 3 inches deep, and $1\frac{3}{4}$ inches in diameter at the mouth, be filled out of a gallon? *Answ.* $115\frac{1}{4}$ times nearly.

RULE III. *To find the content of a frustum of a pyramid:* (1.) To the product of two corresponding sides of the greater and less ends, add one third of the square of their difference; the sum will be the square of a side of the mean base. (2.) From this find the area of the mean base by some of the rules already given for measuring plane surfaces. (3.) Multiply the result by the height.

To find the content of a frustum of a cone: (1.) To the product of the diameters of the two ends, add one third of the square of their difference; the sum will be the square of a mean diameter. (2.) Multiply this square by 0·7854, and the product by the height.

Exam. 2. Given the sides of the bases of a frustum of a regular octagonal pyramid $=19$ and 10 inches respectively, and the length 5 feet 6 inches; required the content.

Here, $19-10=9$, and $19\times10+9^2\div3=217$, the square of a side of the mean base; which being multiplied by 4·8284272, the tabular number (see page 333), we find for the area of the mean base 1047·7687024. The product of this by 5·5, the given length, is 5762·7278632: and dividing this by 144, we obtain 40·01894349 feet, the content required. We divide by 144, instead of 1728, because the length was given in feet.

Exer. 12. If the length of a frustum of a square pyramid be 18 feet 8 inches, the side of its greater base 27 inches, and that of its less 16 inches, what is the content? *Answ.* 61·228395 cubic feet.

13. Required the content of an ale glass in form of the frustum of a cone, the diameter at the mouth being $2\frac{1}{2}$ inches, that of the bottom 1 inch, and the depth 5 inches. *Answ.* 12·76275 cubic inches.

RULE IV. *To find the content of a globe:* Multiply the cube of the diameter by 0·5236.

To find the content of a spheroid: Multiply the fixed axis by the square of the revolving axis and the product by 0·5236.

Exer. 14. Required the contents of three globes, whose diameters are 12, 15, and 21 inches, respectively. *Answ.* 904·7808, 1767·15, and 4849·0596 cubic inches, respectively.

15. Required the content of a balloon in form of a prolate spheroid, having its longest diameter 48 feet, and its shortest 38 feet. *Answ.* 36291·7632 cubic feet.

RULE V. *To find the area of the surface of a body bounded by plane surfaces:* Find the areas of those surfaces separately, and add them together.

To find the area of the curve surface of a right cone: Multiply the circumference of the base by the slant height, and take half the product.

To find the area of the surface of a globe: Multiply the square of the diameter by 3·1416.

Exer. 16. Required the area of the surface of a square pyramid, each side of the base of which is 2 feet 8 inches, and its slant height, measured from the vertex to the middle of any side of the base, 3 feet 9 inches. *Answ.* 27 feet, 1 twelfth, and 4 inches.

17. Required the area of the entire surface of a right cone, the slant height of which is 4 feet 7 inches, and the diameter of its base 2 feet 11 inches. *Answ.* 27·679896 square feet.

18. If the earth were a sphere 7912 miles in diameter, what would be its superficial content? *Answ.* 196,663,355·7504 square miles.

19. Required the superficial contents of the three globes mentioned in Exercise 14. *Answ.* 452·3904, 706·86, and 1385·4456 square inches, respectively.

RULE VI. *To find the content of round or squared timber:* (1.) Take the girt of the tree at the middle, and divide it by 4 (which may be done by halving the line used in taking the girt, and then halving the half thus obtained). (2.) Multiply the square of the quarter girt, thus found, by the length.

If the breadth and depth of squared timber differ considerably, measure them, and multiply their product by the length.

If the tree do not taper uniformly, measure the girts at the middle and ends, or at other equal distances; add the results together, and divide the sum by the number of girts taken; use the quotient as a mean girt, and proceed as before.

If great accuracy were required, timber which tapers uniformly should be measured as the frustum of a pyramid or cone; but when it tapers slowly, as is generally the case, a dimension taken at the middle may be used as a mean without much error. The measurement of round timber, however, by taking, according to the foregoing rule, which is universally adopted in practice, the quarter girt as the side of a square to a mean section of the tree, is very erroneous, giving the content far too small. If it should be wished *to correct the result thus found, it may be done by adding to it one fourth of itself, and to the sum one foot for every* 54 *contained in it.* Another method of approximating the true content is, to *multiply the square of one fifth of the girt by twice the length*; or, which is perhaps easier, *to multiply the square of the girt by* ·08, *and the product by the length.* By both these methods the result is rather too great, and requires to be diminished by one foot in 190.

Exer. 20. Given the mean girt of a square piece of timber = 6 feet 8 inches, and its length = 44 feet 4 inches; required the content. *Answ.* $123{\cdot}14\dot{8}$ cubic feet.

21. Required the content of a round tree, whose length is 32 feet 6 inches, and its mean girt 5 feet 10 inches. *Answ.* 69·119 cubic feet; or, more correctly, 87·9 feet.

22. Given the length of a piece of squared timber = 31 feet 4 inches, and its breadth and depth at the middle = 2 feet 7 inches, and 1 foot 9 inches, respectively; required its content. *Answ.* 141·65 cubic feet.

23. Given the girts of a round tree at the ends, and at two intermediate points equally distant from them and from each other, equal to 10 feet 6 inches, 5 feet 6 inches, 8 feet 8 inches, and 7 feet, respectively, and the length 21 feet 5 inches; required the content. *Answ.* 83·89 cubic feet; or, more correctly, 106·8 feet.

RULE VII. *To find the content of a common cask in imperial gallons:* (1.) Taking the dimensions in inches, to the square of the head diameter add double the square of the bung diameter, and from the sum take four tenths of the square of the difference of those diameters: (2.) Multiply the remainder by the length: (3.) Then multiply the result by 0·0009442.

Instead of multiplying by 0·0009442, we may divide by 1000, and from the quotient, considered as gallons, take one twentieth of itself; the remainder will be the answer, nearly. Should this be large, it may be corrected by rejecting one gallon for 160.

Exer. 24. If the head diameter of a cask be 25 inches, the bung diameter 34 inches, and the length 43 inches, how many gallons does it contain? *Answ.* 118 gallons, nearly.

Exer. 25. Given the length of a rectangular field = 15 perches; required its breadth so that it may contain an acre. *Answ.* $10\frac{2}{3}$ perches.

26. If a horse be bound in the middle of a field by a cord, one end of which is fixed at a certain point; what must be the length of the cord, that the horse may be allowed to graze on exactly an acre? *Answ.* 7·1365 perches.

27. Given the height of a stone column in form of a frustum of a cone = 28 feet 6 inches, and the diameters of its ends 3 feet, and 2 feet 3 inches, respectively; required its weight, a cubic foot of the stone weighing 2568 ounces, avoirdupois. *Answ.* 11 tons, 2 cwt., 2 qrs., 3 lbs.

28. Required the diameter of a circle whose area is a square foot. *Answ.* 13·54054 inches.

29. Required the diameter of a globe whose content is a cubic foot. *Answ.* 14·8884 inches.

30. Given the diameter of the base of a cone = 50 inches, and its content = 50 cubic feet; required its height. *Answ.* 11 feet, nearly.

31. How many yards of carpeting, 27 inches wide, will cover a rectangular floor, 22 feet 6 inches long, and 16 feet 8 inches broad? *Answ.* 55 yards, 1 foot, 8 inches.

32. How much ground is occupied by 100 miles of a road, 63 feet in width? *Answ.* $763\frac{7}{11}$ acres.

33. If a cubic foot of water weigh 1000 ounces, and a cubic foot of cast iron 7000 ounces, what is the weight of a cylindrical cast-iron pipe, 100 feet in length, a quarter of an inch in thickness, and having its interior diameter 3 inches? Find, also, the weight of the water which it can contain. *Answ.* 6 cwt. 3 qrs. $19\frac{1}{2}$ lbs., nearly, and 2 cwt. 2 qrs. 27 lbs., nearly.

34. If the length of a cask be 45 inches, and its head and bung diameters 27 and 37 inches, respectively, what weight of pure water will it contain? *Answ.* 13 cwt.

35. If the cylinder of a steam engine be 3 feet in diameter, and 5 feet deep, how much steam can it contain? *Answ.* 35·343 cubic feet.

TABLE I., SHOWING THE AMOUNT OF £1, AT COMPOUND INTEREST.

Yrs.	3 per cent.	4 per cent.	5 per cent.	6 per cent.
1	1·030,000	1·040,000	1·050,000	1·060,000
2	1·060,900	1·081,600	1·102,500	1·123,600
3	1·092,727	1·124,864	1·157,625	1·191,016
4	1·125,509	1·169,859	1·215,506	1·262,477
5	1·159,274	1·216,653	1·276,282	1·338,226
6	1·194,052	1·265,319	1·340,096	1·418,519
7	1·229,874	1·315,932	1·407,100	1·503,630
8	1·266,770	1·368,569	1·477,455	1·593,848
9	1·304,773	1·423,312	1·551,328	1·689,479
10	1·343,916	1·480,244	1·628,895	1·790,848
11	1·384,234	1·539,454	1·710,339	1·898,299
12	1·425,761	1·601,032	1·795,856	2·012,196
13	1·468,534	1·665,074	1·885,649	2·132,928
14	1·512,590	1·731,676	1·979,932	2·260,904
15	1·557,967	1·800,944	2·078,928	2·396,558
16	1·604,706	1·872,981	2·182,875	2·540,352
17	1·652,848	1·947,900	2·292,018	2·692,773
18	1·702,433	2·025,817	2·406,619	2·854,339
19	1·753,506	2·106,849	2·526,950	3·025,600
20	1·806,111	2·191,123	2·653,298	3·207,135
21	1·860,295	2·278,768	2·785,963	3·399,564
22	1·916,103	2·369,919	2·925,261	3·603,537
23	1·973,587	2·464,716	3·071,524	3·819,750
24	2·032,794	2·563,304	3·225,100	4·048,935
25	2·093,778	2·665,836	3·386,355	4·291,871
26	2·156,592	2·772,470	3·555,673	4·549,383
27	2·221,289	2·883,369	3·733,456	4·822,346
28	2·287,928	2·998,703	3·920,129	5·111,687
29	2·356,566	3·118,651	4·116,136	5·418,388
30	2·427,262	3·243,398	4·321,942	5·743,491
31	2·500,080	3·373,133	4·538,039	6·088,101
32	2·575,083	3·508,059	4·764,941	6·453,386
33	2·652,335	3·648,381	5·003,189	6·840,590
34	2·731,905	3·794,316	5·253,348	7·251,025
35	2·813,862	3·946,089	5·516,015	7·686,087
36	2·898,278	4·103,933	5·791,816	8·147,252
37	2·985,227	4·268,090	6·081,407	8·636,087
38	3·074,783	4·438,813	6·385,477	9·154,252
39	3·167,027	4·616,366	6·704,751	9·703,507
40	3·262,038	4·801,021	7·039,989	10·285,718
41	3·359,899	4·993,061	7·391,988	10·902,861
42	3·460,696	5·192,784	7·761,588	11·557,033
43	3·564,517	5·400,495	8·149,667	12·250,455
44	3·671,452	5·616,515	8·557,150	12·935,482
45	3·781,596	5·841,176	8·985,008	13·764,611
46	3·895,044	6·074,823	9·434,258	14·590,487
47	4·011,895	6·317,816	9·905,971	15·465,917
48	4·132,252	6·570,528	10·401,270	16·393,872
49	4·256,219	6·833,349	10·921,333	17·377,504
50	4·383,906	7·106,683	11·467,400	18·420,154

TABLE II., SHOWING THE AMOUNT OF AN ANNUITY OF £1

Yrs.	3 per cent.	4 per cent.	5 per cent.	6 per cent.
1	1·000,000	1·000,000	1·000,000	1·000,000
2	2·030,000	2·040,000	2·050,000	2·060,000
3	3·090,900	3·121,600	3·152,500	3·183,600
4	4·183,627	4·246,464	4·310,125	4·374,616
5	5·309,135	5·416,322	5·525,631	5·637,092
6	6·468,409	6·632,975	6·801,912	6·975,318
7	7·662,462	7·898,294	8·142,008	8·393,837
8	8·892,336	9·214,226	9·549,108	9·897,467
9	10·159,106	10·582,795	11·026,564	11·491,315
10	11·463,879	12·006,107	12·577,892	13·180,794
11	12·807,795	13·486,351	14·206,787	14·971,642
12	14·192,029	15·025,805	15·917,126	16·869,941
13	15·617,790	16·626,837	17·712,982	18·882,137
14	17·086,324	18·291,911	19·598,631	21·015,065
15	18·598,913	20·023,587	21·578,563	23·275,969
16	20·156,881	21·824,531	23·657,491	25·672,528
17	21·761,587	23·697,512	25 840,366	28·212,879
18	23·414,435	25·645,412	28·132,384	30·905,652
19	25·116,868	27·671,229	30·539,003	33 759,991
20	26·870,374	29·778,078	33·065,954	36·785,591
21	28·676,485	31·969,201	35·719,251	39·992,726
22	30 536,780	34·247,969	38·505,214	43·392,290
23	32·452,883	36·717,888	41·430,475	46·995,827
24	34·426,470	39·082,604	44·501,998	50·815,577
25	36·459,264	41·645,908	47·727,098	54·864,512
26	38·553,042	44·311,744	51·113,453	59·156,382
27	40·709,633	47·084,214	54·669,126	63·705,765
28	42 930,922	49·967,582	58·402,582	68·528,111
29	45·218,850	52·966,286	62·322,711	73·639,798
30	47·575,415	56·084,937	66·438,847	79·058,186
31	50·002,678	59·328,335	70·760,789	84·801,677
32	52·502,758	62·701,468	75·298,289	90·889,778
33	55·077,841	66·209,527	80·063,770	97·343,164
34	57·730,176	69·857,908	85·066,959	104·183,754
35	60·462,081	73·652,224	90·320,307	111·434,779
36	63·275,944	77·598,313	95·836,322	119·120,866
37	66·174,222	81·702,246	101·628,138	127·268,118
38	69·159,449	85·970,336	107·709,545	135·904,205
39	72·234,232	90·409,149	114·095,023	145·058,458
40	75·401,259	95·025,515	120·799,774	154·761,965
41	78·663,297	99·826,536	127·839,762	165 047,683
42	82·023,196	104·819,597	135·231,751	175·950,544
43	85·483,892	110·012,381	142·993,338	187·507,577
44	89·048,409	115·412,876	151·143,005	199·758,031
45	92·719,861	121·029,392	159·700 155	212·743,513
46	96·501,457	126·870,567	168·685,163	226·508,124
47	100·396,500	132·945,390	178·119,421	241·098,612
48	104·408,395	139·263,206	188·025,392	256·564,528
49	108·540,647	145·833,734	198·426,662	272·958,400
50	112·796,867	152·667,083	209·347,995	290·335,904

TABLE III., SHOWING THE PRESENT VALUE OF AN ANNUITY OF £1.

Yrs.	3 per cent.	4 per cent.	5 per cent.	6 per cent.
1	·970,874	·961,538	·952,381	·943,396
2	1·913,470	1·886,194	1·859,410	1·833,392
3	2·828,612	2·775,190	2·723,248	2·673,011
4	3·717,099	3·629,994	3·545,950	3 465,105
5	4·579,708	4·451,821	4·329,476	4·212,363
6	5·417,192	5·242,136	5·075,691	4·917,324
7	6·230,284	6·002,054	5·786,372	5·582,381
8	7·019,693	6·732,744	6·463,211	6·209,793
9	7·786,110	7·435,331	7·107,820	6·801,691
10	8·530,204	8·110,895	7 721,733	7·360,086
11	9·252,625	8·760,576	8·306,412	7·886,874
12	9·954,005	9·385,073	8·863,249	8·383,843
13	0·634,956	9·985,647	9 393,570	8 852,682
14	11·296,074	10·563,122	9 898,638	9·294,983
15	11·937,936	11·118,487	10 379.655	9·712,248
16	12·561,103	11·652,395	10·837,767	10·105,894
17	13·166,119	12·165,668	11·274 064	10·477,258
18	13·753,514	12·659,396	11·689,585	10·827,602
19	14·323,800	13·133,938	12·085,319	11·158,115
20	14·877,476	13·590,325	12·462,208	11·469,920
21	15·415,025	14·029,159	12·821,150	11·764,075
22	15·936,918	14·451,114	13·163,000	12·041,580
23	16·443,610	14·856,840	13·488,571	12·303,377
24	16·935,544	15·246,961	13·798,639	12·550,356
25	17·413,150	15·622,078	14·093,942	12·783,355
26	17·876,845	15·982,767	14·375,183	13·003,165
27	18·327,034	16·329,584	14·643,031	13·210,533
28	18·764,111	16·663,961	14·898,125	13·406,163
29	19·188,457	16·983,712	15·141,071	13·590,720
30	19·600,444	17·292,031	11·372,448	13·764,830
31	20·000,431	17·588,491	15·592,807	13·929,085
32	20·388,768	17·873,549	15·802,673	14·084,042
33	20·765,794	18·147,643	16·002,546	14·230,228
34	21·131,839	18·411,195	16·192,901	14·368,140
35	21·487,222	18·664,610	16·374,191	14·498,245
36	21·832,254	18·908,279	16·546,848	14·620,986
37	22·167,237	19·142,576	16·711,284	14·736,779
38	22·492,463	19·367,861	16·867,889	14·846,018
39	22·808,217	19·584,482	17·017,037	14·949,074
40	23·114,774	19·792,771	17·159,083	15·046,296
41	23·412,402	19·993,049	17·294,365	15·138,015
42	23·701,361	20·185,624	17·423,205	15·224,542
43	23·981,904	20·370,792	17·545,909	15·306,172
44	24·254,276	20·548,838	17·662,770	15·383,181
45	24·518,715	20·720,036	17·774,067	15·455,831
46	24·775,452	20·884,650	17·880,064	15·524,369
47	25·024,711	21·042,933	17·981,013	15·589,027
48	25·266,710	21·195,128	18·077,155	15·650,025
49	25·501,660	21·341,469	18·168,719	15·707,571
50	25·729,767	21·482,182	18·255,923	15·761,859

TABLE IV., SHOWING THE VALUE OF AN ANNUITY ON A SINGLE LIFE.

Ages	3 per cent.	4 per cent.	5 per cent.	Ages	3 per cent.	4 per cent.	5 per cent.
10	24·148	20·077	17·057	54	12·462	11·351	10·396
11	23·995	19·982	16·998	55	12·094	11·043	10·135
12	23·814	19·865	16·919	56	11·724	10·731	9·871
13	23·610	19·728	16·824	57	11·353	10·417	9·602
14	23·390	19·578	16·717	58	10·981	10·100	9·330
15	23·158	19·417	16·602	59	10·608	9·780	9·054
16	22·922	19·252	16·482	60	10·236	9·459	8·776
17	22·686	19·087	16·362	61	9·866	9·138	8·497
18	22·458	18·928	16·248	62	9·498	8·818	8·217
19	22·243	18·780	16·142	63	9·134	8·500	7·938
20	22·043	18·644	16·047	64	8·774	8·185	7·659
21	21·848	18·513	15·957	65	8·418	7·870	7·381
22	21·656	18·384	15·868	66	8·064	7·557	7·102
23	21·460	18·251	15·776	67	7·712	7·243	6·821
24	21·254	18·110	15·678	68	7·360	6·928	6·538
25	21·038	17·961	15·572	69	7·007	6·610	6·251
26	20·814	17·804	15·460	70	6·657	6·293	5·968
27	20·582	17·641	15·342	71	6·311	5·979	5·676
28	20·347	17·474	15·221	72	5·975	5·672	5·395
29	20·109	17·304	15·097	73	5·653	5·377	5·123
30	19·867	17·131	14·971	74	5·348	5·097	4·866
31	19·623	16·955	14·842	75	5·061	4·833	4·622
32	19·373	16·774	14·708	76	4·782	4·574	4·382
33	19·117	16·587	14·570	77	4·512	4·324	4·149
34	18·855	16·395	14·426	78	4·249	4·079	3·921
35	18·587	16·197	14·277	79	3·992	3·838	3·695
36	18·314	15·994	14·124	80	3·742	3·604	3·475
37	18·037	15·786	13·966	81	3·507	3·382	3·266
38	17·756	15·575	13·805	82	3·290	3·178	3·073
39	17·469	15·358	13·638	83	3·089	2·989	2·894
40	17·176	15·135	13·466	84	2·908	2·818	2·732
41	16·876	14·904	13·287	85	2·739	2·658	2·581
42	16·566	14·664	13·099	86	2·570	2·498	2·430
43	16·248	14·417	12·903	87	2·393	2·330	2·270
44	15·924	14·162	12·701	88	2·206	2·152	2·100
45	15·594	13·901	12·491	89	1·987	1·942	1·898
46	15·260	13·635	12·278	90	1·740	1·704	1·669
47	14·923	13·366	12·061	91	1·487	1·459	1·432
48	14·585	13·094	11·840	92	1·229	1·208	1·188
49	14·242	12·817	11·614	93	·951	·937	·924
50	13·896	12·536	11·383	94	·677	·668	·660
51	13·545	12·249	11·146	95	·415	·411	·406
52	13·188	11·955	10·902	96	·178	·177	·175
53	12·826	11·655	10·651	97	·000	·000	·000

TABLE V., SHOWING THE PRESENT VALUE OF AN ANNUITY OF £1 ON THE JOINT CONTINUANCE OF TWO LIVES.

Ages		3 per cent.	4 per cent.
10	10	21·008	17·866
	15	20·405	17·435
	20	19·658	16·888
	25	18·979	16·411
	30	18·122	15·786
	35	17·133	15·051
	40	15·991	14·181
	45	14·657	13·130
	50	13·180	11·934
	55	11·568	10·591
	60	9·867	9·135
	65	8·171	7·649
	70	6·500	6·150
	75	4·966	4·744
	80	3·686	3·551
15	15	19·866	17·044
	20	19·187	16·539
	25	18·571	16·101
	30	17·774	15·516
	35	16·841	14·819
	40	15·750	13·985
	45	14·462	12·968
	50	13·027	11·805
	55	11·452	10·492
	60	9·784	9·063
	65	8·116	7·601
	70	6·468	6·121
	75	4·950	4·730
	80	3·680	3·546
20	20	18·582	16·081
	25	18·039	15·689
	30	17·315	15·153
	35	16·451	14·504
	40	15·424	13·714
	45	14·194	12·740
	50	12·809	11·615
	55	11·279	10·338
	60	9·650	8·942
	65	8·015	7·508
	70	6·394	6·053
	75	4·899	4·682
	80	3·646	3·513
25	25	17·570	15·346
	30	16·926	14·862
	35	16·139	14·265
	40	15·182	13·524
	45	14·013	12·595
	50	12·679	11·508
	55	11·189	10·262
	60	9·590	8·890
	65	7·977	7·475
	70	6·373	6·033
	75	4·888	4·671
	80	3·640	3·508
30	30	16·373	14·440
	35	15·681	13·908
	40	14·816	13·232
	45	13·731	12·365
	50	12·469	11·332
	55	11·038	10·132
	60	9·486	8·798
	65	7·907	7·412
	70	6·328	5·992
	75	4·860	4·645
	80	3·623	3·492

Ages		3 per cent.	4 per cent.
35	35	15·095	13·450
	40	14·341	12·854
	45	13·363	12·064
	50	12·195	11·104
	55	10·844	9·967
	60	9·354	8·683
	65	7·821	7·335
	70	6·274	5·944
	75	4·828	4·616
	80	3·605	3·475
40	40	13·710	12·348
	45	12·862	11·656
	50	11·818	10·789
	55	10·573	9·737
	60	9·171	8·524
	65	7·703	7·230
	70	6·203	5·879
	75	4·787	4·578
	80	3·582	3·453
45	45	12·162	11·076
	50	11·269	10·327
	55	10·166	9·387
	60	8·886	8·275
	65	7·515	7·062
	70	6·085	5·772
	75	4·717	4·513
	80	3·543	3·416
50	50	10·543	9·710
	55	9·611	8·908
	60	8·486	7·924
	65	7·245	6·820
	70	5·915	5·617
	75	4·615	4·419
	80	3·484	3·361
55	55	8·868	8·260
	60	7·931	7·433
	65	6·856	6·471
	70	5·663	5·387
	75	4·462	4·276
	80	3·395	3·277
60	60	7·199	6·779
	65	6·321	5·987
	70	5·301	5·055
	75	4·233	4·064
	80	3·258	3·148
65	65	5·652	5·377
	70	4·831	4·621
	75	3·927	3·778
	80	3·069	2·969
70	70	4·223	4·056
	75	3·510	3·387
	80	2·801	2·716
75	75	2·988	2·893
	80	2·442	2·374
80	80	2·049	1·997

ANSWERS TO EXERCISES.

NUMERATION.

Exer. 1. Twenty-four. 2. One hundred and forty-four. 3. Three hundred and sixty-five. 4. One thousand. 5. One thousand, seven hundred and twenty-eight. 6. Two thousand, two hundred and forty. 7. Nine thousand, seven hundred and ninety. 8. Thirty-seven thousand, and forty-eight. 9. Thirty thousand and nine. 10. Four millions, fifty-five thousand, and seventy. 11. Three hundred thousand, four hundred and five. 12. Seventy-nine millions, five hundred and three thousand, and forty-six. 13. Eight hundred millions, five hundred and sixty thousand, and eighty. 14. Fifty-seven millions, two hundred and ninety thousand. 15. Six hundred and eighty millions, and forty-two. 16. Ninety-three millions, ninety thousand, and ninety-three. 17. One hundred and thirteen thousand, three hundred and fifty-five. 18. Seven hundred and eighty-five thousand, three hundred and ninety-eight. 19. Seven millions, thirty thousand, four hundred and sixty-two. 20. Twenty-four millions, nine hundred and two thousand, four hundred and ninety. 21.* Nine billions, three millions, eight thousand and five. 22.* One hundred billions, and one thousand. 23.* Sixty billions, six hundred and sixty millions, six hundred and seven thousand, and seven. 24.* One trillion, twenty billions, three hundred and four millions, fifty thousand, six hundred and seven. 25.* Nine hundred and ten billions, one hundred and ten millions, one hundred and twenty thousand, three hundred and one. 26.* Two hundred billions, thirty millions, forty thousand, five hundred and thirty-eight. 27.* Eight hundred and twenty trillions, seven hundred and sixty billions, five millions, one hundred and ninety-two thousand, six hundred and forty-five. 28.* Ten trillions, ten billions, one million, one hundred. 29.* Forty trillions, five hundred and six billions, seventy millions, eighty-nine thousand. 30.* Seven hundred and ninety-four quadrillions, six hundred and twenty-eight trillions, nine hundred billions, six hundred and forty millions, thirty thousand, and four.

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* *According to the common Notation:*

Exer. 21. Nine thousand and three millions, eight thousand and five. 22. One hundred thousand millions, and one thousand. 23. Sixty thousand six hundred and sixty millions, six hundred and seven thousand, and seven. 24. One billion, twenty thousand three hundred and four millions, fifty thousand, six hundred and seven. 25. Nine hundred and ten thousand, one hundred and ten millions, one hundred and twenty thousand, three hundred and one. 26. Two hundred thousand and thirty millions, forty thousand, five hundred and thirty-eight. 27. Eight hundred and twenty billions, seven hundred and sixty thousand and five millions, one hundred and ninety-two thousand, six hundred and forty-five. 28. Ten billions, ten thousand and one millions, one hundred. 29. Forty billions, five hundred and six thousand and seventy millions, eighty-nine thousand. 30. Seven hundred and ninety-four thousand six hundred and twenty-eight billions, nine hundred thousand six hundred and forty millions, thirty thousand and four.

NOTATION.

Ex. 1. 52
2. 300
3. 504
4. 1024
5. 2000
6. 1815
7. 7854

Ex. 8. 3008
9. 5070
10. 4504
11. 20084
12. 650090
13. 7007010
14. 64000300

Ex. 15. 11002000
16. 110020000
17. 1050000
18. 1000200000*
19. 70000010088*
20. 900068000020*

21. 35500000, 66300000, 91600000, 140000000, 476000000, 874000000, 1760000000, 2750000000.

SIMPLE ADDITION.

Ex. 1. 21555
2. 206343
3. 1508939
4. 1383458
5. 21225092
6. 904388
7. 121888988
8. 27457989
9. 86171735
10. 20967445

Ex. 11. 463140294
12. 444470727
13. 10867114613
14. 96840996
15. 207305
16. In 1819 or 1820
17. In 1399
18. 36633 miles.

SIMPLE SUBTRACTION.

Ex. 1. 13031
2. 15708
3. 17368
4. 32131
5. 590731
6. 8285367
7. 3209877
8. 763544529
9. 10101001
10. 333232333
11. 7468053687
12. 327504427
13. 99579930

Ex. 14. 89998999
15. 28 yards
16. 2997000
17. 164 gallons
18. 299997
19. 78 years
20. 5720 feet

Ex. 21.

1253	273
884	261
779	232
660	221
622	189
584	74
545	70
463	60
435	46
425	37
422	29
383	20
332	16
287	10
285	9
284	

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* *According to the common Notation:*

Ex. 18. 1000000200000
19. 70000000010088
20. 900000068000020

SIMPLE MULTIPLICATION.

Ex. 1. 432	Ex. 6. 1000	Ex. 11. 1615040
2. 3180	7. 27414	12. 161504000
3. 3645	8. 80388	13. 573440000
4. 5616	9. 20748	14. 7068600
5. 11193	10. 18160	15. 49152000

SIMPLE DIVISION.

Ex. 1. 156950	Ex. 4. $630341\frac{1}{6}$	Ex. 7. $82882024\frac{4}{9}$
2. $457441\frac{1}{4}$	5. $1023648\frac{4}{7}$	8. $4840788\frac{7}{11}$
3. $853196\frac{3}{5}$	6. $3432416\frac{4}{8}$	9. $4106265\frac{9}{12}$

COMPOUND ADDITION.

	£	s.	d.
Ex. 1.	3852	15	10
2.	36	12	$3\frac{1}{4}$
3.	21849	18	$2\frac{1}{4}$
4.	48	0	$6\frac{1}{4}$
5.	2431	9	$2\frac{3}{4}$
6.	3765	15	$1\frac{1}{8}$
7.	1478	13	$1\frac{1}{2}$
8.	13512	6	9
9.	1503	4	$6\frac{1}{4}$
10.	10	4	6
11.	2204	10	$4\frac{3}{4}$
12.	313	12	$6\frac{1}{2}$
13.	515	1	1
14.	4488	5	$7\frac{1}{4}$

	£	s.	d.
Ex. 15.	2162	14	$0\frac{1}{4}$
16.	3555	12	$8\frac{1}{4}$
17.	4600	6	4

18. 224 cwt. 1 qr. 11 ℔s.
19. 97 ℔s. 1 oz. 15 drs.
20. 383 tons, 8 cwt.
21. 362 ℔s.
22. 552 cwt. 0 q. 5 ℔s.
23. 579 cwt. 3 q. 9 ℔s.
24. 306 t. 12 c. 2 q. 17 ℔s.
25. 489 a. 1 r. 23 p.
26. 1115 a. 1 r. 12 p.
27. 42 ft. 2 in.
28. 219° 32′.

COMPOUND SUBTRACTION.

	£	s.	d.
Ex. 1.	10	8	$6\frac{1}{4}$
2.	451	1	$9\frac{1}{2}$
3.	103	14	7
4.	263	12	$7\frac{3}{4}$
5.	7	2	$4\frac{1}{2}$
6.	98	17	3
7.	2	19	5
8.	2	13	0

	c.	q.	℔s.
9.	4	1	8
10.	5	1	11
11.	25	2	15
12.	9 t. 14	3	14

	£	s.	d.
13.	0	0	7
14.	0	0	11

Ex. 15.	6°	57′	2″
	2	39	43
	1	52	24
	2	34	44
	3	58	26
16.	57	45	0
17.	22	3	15
18.	1	16	0

	d.	h.	m.
19.	136	17	33
	140	13	20
	321	17	22
	3645	14	31
	6426	15	15
	19927	14	25

COMPOUND MULTIPLICATION.

	£	s.	d.		£	s.	d.
Ex. 1.	7	9	7	Ex. 7.	23	10	0
2.	5	13	$2\frac{1}{4}$	8.	60	0	0
3.	3	15	9	9.	7	18	$6\frac{1}{2}$
4.	7	11	8	10.	103	4	4
5.	22	16	$1\frac{1}{2}$	11.	47	15	6
6.	7	19	3	12.	21	19	0

COMPOUND DIVISION.

	£	s.	d.	Rem.		£	s.	d.	Rem.
Ex. 1.	5	5	$10\frac{1}{4}$		Ex. 7.	0	9	$5\frac{1}{4}$	...1*d.*
2.	0	18	$10\frac{1}{4}$	...$\frac{1}{4}$*d.*	8.	0	13	$1\frac{1}{4}$	...$\frac{1}{4}$*d.*
3.	1	1	$5\frac{1}{4}$	...$\frac{3}{4}$*d.*	9.	0	10	3	
4.	1	17	$10\frac{3}{4}$	...$\frac{1}{2}$*d.*	10.	1	13	$5\frac{1}{4}$	
5.	1	8	$0\frac{1}{2}$	...1*d.*	11.	0	3	4	
6.	0	5	$7\frac{1}{2}$	...1*d.*	12.	0	2	$10\frac{1}{4}$	...$\frac{1}{4}$*d.*

NOTE.

On Horner's Method of Resolving Equations.

The rules for the extraction of roots given in pages 260 and 264,* are only particular applications of the general method of resolving algebraic equations, discovered by the late Mr. Horner of Bath. As that method—the best by far that has been given for the purpose—is but little known, the following brief article may not be unacceptable to some mathematical readers, especially those who may not have time to study the subject at greater length.

The rules for evolution, already referred to, will serve equally for the resolution of numerical equations, if the columns, instead of being headed with ciphers, be headed with the coefficients of the given equation. Thus, if it be required to resolve the equation,

$$x^3 + 3x^2 + 2x = 71, \text{ or } x^3 + 3x^2 + 2x - 71 = 0,\dagger$$

the work will be as follows, one of the roots being found by trial to lie between 3 and 4 :—

3	2	−71(3·2213072, *answ.*
3	18	60
6	20	−11000
3	27	9888
9	4700	−1112000
3	244	1043448
120	4944	−68552
2	248	52438
122	519200	−16114
2	2524	15735
124	521724	−379
2	2528	367
1260	524252	−12
2	13·	10
1262	52438	−2
2	13	
1264	52451	
2	....	
1266		
..		

* The common method of extracting the square root (page 256) is also, virtually, a case of the same general method.

† For obtaining uniformity in the process, it is better to have all the significant terms in the same member, and then to take as headings for the columns in the work the various numbers, with their respective signs. In that case, all the

If the equation to be resolved were $x^3+5x=50$, or, which is equivalent, $x^3+0x^2+5x-50=0$, the headings would be 0, 5, and -50; while, if it were $x^3+5x^2=50$, or $x^3+5x^2+0x-50=0$, they would be 5, 0, and -50. If, again, the equation were $x^3-7x^2-3x-6=0$, the headings would be -7, -3, and -6.

As another example, let it be required to resolve the equation, $x^4+x^3-29x^2-27x-6=0$. A few trials will show, that one value of x lies between 5 and 6; the substitution of 5 for x giving $5^4+5^3-29\times5^2-27\times5-6$, or -116, instead of 0; while the substitution of 6 gives, instead of 0, $6^4+6^3-29\times6^2-27\times6-6$, or 300: and these results having opposite signs, at least one value of x must lie between 5 and 6.* Of this value, therefore, the first figure is 5; and the work will be as follows:—

1	−29	−27	−6(5·372282, *answ.*
5	30	5	−110
6	1	−22	−1160000
5	55	280	902151
11	56	258000	−257849
5	80	42717	249459
16	13600	300717	−8390
5	639	44661	7354
210	14239	345378	−1036
3	648	1099·	736
213	14887	35637	−300
3	657	1106	294
216	15544	36743	−6
3	1·	3	7
219	157	3677	
3	1	3	
222	158	3680	
•	1	••	
	159		
	•		

---

products by each figure of the root, must be *added*, in the *algebraic* sense; though when the signs are different, the process comes to be *subtraction* in the *arithmetical* sense. Thus, in adding 60 and -71, we get -11, as the algebraic sum. It may also be remarked, that, in this example, -71 may be regarded as the coefficient of x^0, which is equivalent to 1.

* Such trials are effected most easily by employing the same sort of process, as that which is used at the commencement of the work for finding the root. Thus,

	1	−29	−27	−6(5
	5	30	5	−110
	6	1	−22	−116
Also,	1	−29	−27	−6(6
	6	42	78	306
	7	13	51	300

Here, the last results, -116 and 300, are the same as those found by the other method.

By farther trial, it would be found, that another of the roots lies between -5 and -6, and the work for finding it will be as follows:—

1	−29	−27	−6(−5·449489
−5	20	45	−90
−4	−9	18	−960000
−5	45	−180	830016
−9	36	−162000	−129984
−5	70	−45504	104568
−14	10600	−207504	−25416
−5	776	−48672	24102
−190	11376	−256176	−1314
−4	792	−524·	1076
−194	12168	−26142	−238
−4	808	−524	215
−198	12976	−26666	−23
−4	1·	−12	24
−202	131	−2678	
−4	•	−11	
−206		−2689	
•		•••	

The former of these two roots should be 5·3722813, and the latter −5·4494897; so that, by short and easy operations, they have both been determined with much accuracy: and by postponing sufficiently the commencement of the contracted part of the process, any assigned degree of accuracy whatever might be attained. It would be found in a similar manner, that the two remaining roots are −0·3722813 and −0·5505103.*

To illustrate the principle of this method in a familiar manner, let us assume the equation,

$$x^3+ax^2+bx+c=0\ldots\ldots\ldots\ldots\ldots(1);$$

* The mathematical reader will recollect, that after finding the first root, we might divide $x^4+x^3-29x^2-27x-6$ by $x-5{\cdot}372282$; and that, by putting the quotient equal to 0 we should have a cubic equation, the three roots of which would be the remaining roots of the given equation: and again, that after one of the roots of that cubic equation was computed, a similar division would give a quadratic equation, the roots of which would be the two remaining roots of the original equation.

The division by $x-5{\cdot}372282$ is most easily effected by a process of exactly the same nature, as that employed in the first line of any of the operations for finding roots; the sole difference being, that 5·372282 is employed, instead of a single figure. The work is as follows:—

1	−29	−27	−6(5·372282
5·372282	34·233696	28·116891	6·000253
6·372282	5·233696	1·116891	0·000253

In this process 34·233696 is the product of 5·372282 and 6·372282; 28·116891, that of 5·372282 and 5·233696; and 6·000253, that of 5·372282 and 1·116891. Were 5·372282 the exact root, there would be no remainder. The smallness, however, of the remainder, 0·000253, shows, that 5·372282 must be a near approximation to one of the roots. Employing the first three numbers now obtained, we find that the cubic equation above referred to, is $x^3+6{\cdot}372282x^2+5{\cdot}233696x+1{\cdot}116891=0$.

and let us suppose that r is a part of the value of x (the first figure in the present case). Let also x' be the remaining part of x so that $x=r+x'$. Hence,

$$x^2=r^2+2rx'+x'^2, \text{ and } x^3=r^3+3r^2x'+3rx'^2+x'^3;$$

the substitution of which in equation (1) gives

$$r^3+3r^2x'+3rx'^2+x'^3+ar^2+2arx'+ax'^2+br+bx'+c=0, \text{ or}$$
$$x'^3+(3r+a)x'^2+(3r^2+2ar+b)x'+r^3+ar^2+br+c=0 \ldots (2).*$$

The resolution of this equation would give the value of x', the part of x still to be found: and the following process will show, that the coefficients of equation (2) will be obtained by applying Horner's process to those of equation (1):—

$$\begin{array}{lll}
a & b & c(r \\
\underline{r} & \underline{r^2+\ ar} & \underline{r^3+ar^2+br} \\
r+a & r^2+\ ar+b & r^3+ar^2+br+c \\
\underline{r} & \underline{2r^2+\ ar} & \\
2r+a & 3r^2+2ar+b & \\
\underline{r} & & \\
3r+a & &
\end{array}$$

Here we add r in the first column. Then, multiplying the sum, $r+a$, by r, we write the result, r^2+ar, in the second column. Hence, by addition we get r^2+ar+b; the product of which by r is set in the third column, and added to c. Then, returning to the first column, we add, multiply, &c., according to Horner's process; and the final results turn out to be the same as the coefficients of x'^2, x', and x'^0, in equation (2): and a like illustration may be given in every case.

If we now put, for simplicity and brevity, a', b', c', to denote the three results just found, equation (2) becomes

$$x'^3+a'x'^2+b'x'+c'=0 \ldots\ldots\ldots (3).$$

This equation exactly resembles equation (1); and therefore, if the first figure in the value of x', which is plainly the second figure in that of x, be denoted by r', and the remaining part of the value by x'', so that $x'=r'+x''$, an equation of the form

$$x''^3+a''x''^2+b''x''+c''=0,$$

would be obtained by the usual process. From this, in a similar manner, the first figure of the value of x'', or a third in that of x, would be found: and thus the process might be continued, till there should be no remainder, or till as many figures were computed, as should be considered necessary.

* It will readily appear to the mathematical reader, that in this and every similar equation found by changing x into $x'+r$, the coefficients of the powers of x' taken in reversed order, are the same as the first member of the original equation, its first differential coefficient divided by 1, its second divided by 1×2, its third by $1\times 2\times 3$, &c., x being changed into r in each of them.

From (3), by transposing c', and dividing by $x'^2+a'x'+b'$, we get

$$x' = -\frac{c'}{x'^2+a'x'+b'};$$

which, when x' is small, becomes nearly

$$x' = -\frac{c'}{b'}:$$

and hence we see the reason of dividing the number in the last column by the one in the column before it, with the contrary sign, to find the figure to be tried as the next figure of the root.

The principle of the method of contraction, which is evidently of great importance, will be easily understood by those who are acquainted with the contracted method of division in decimal fractions. It may be illustrated by working an example at full length, and then cutting off the superfluous figures. The annexing of one cipher in the first column, two in the second, &c., has the advantage of preventing all necessity of employing the separating point for decimals, and of considering its position, except in the root, where it is to be placed, as soon as all the periods of whole numbers in the number heading the last column, have been employed.

As the extraction of the cube root is evidently a particular case of the general method, its explanation is contained in what is given above. It will be the same, in fact, if a and b be taken each equal to nothing, and c negative in equations (1) and (2).

It may be remarked, that this method of extracting the cube root is, in substance, the same as that which is given in most of the books on arithmetic. The *mode of operating*, however, is in some degree changed for the purpose of rendering the process more simple and easy. When presented thus, too, it forms an easy and an appropriate introduction to the method of resolving equations, above illustrated.

It may also be remarked in conclusion, that in addition to the one important method of contraction which has been given in what precedes, various other abbreviations in the arithmetical processes will readily suggest themselves to the intelligent reader. Few of these, however, will be found to be of much value; as, in most instances, they do little more than shorten the process to the eye, without abridging the mental labour.

The experienced algebraist will prefer the following investigation of this method of solution.

Let $x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n=0$, be the equation to be resolved; and let the first member be represented by fx. Now, by the theory of equations, if r be a root of $fx=0$, fx will be divisible by $x-r$. Let r not be a root, however, and in that case, there will be a remainder, so that

$$\frac{fx}{x-r}=f_1x+\frac{R_1}{x-r}; \text{ and, consequently, } fx=f_1x(x-r)+R_1\ldots(1):$$

where f_1x is an expression of the $(n-1)$th degree in x, such as $x^{n-1}+p'_1x^{n-2}+$ &c. In a similar manner we should get

$$f_1x=f_2x.(x-r)+R_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$
$$f_2x=f_3x.(x-r)+R_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$f_{n-2}x=f_{n-1}x(x-r)+R_{n-1} \ldots\ldots\ldots (n-1)$$

where f_2x, f_3x, ..., and $f_{n-1}x$ are expressions of the $(n-2)$th, $(n-3)$th, ..., and 1st orders, while R_1, R_2, ..., and R_{n-1} are mere numbers. Now, since the coefficient of x^n in fx is unity, it follows from the nature of division, that the coefficients of the highest powers of x in the expressions, f_1x, f_2x, ..., and $f_{n-1}x$, are each unity. Hence, $f_{n-1}x$ must be of the form $x+A$; and therefore we have

$$\frac{f_{n-1}x}{x-r}=1+\frac{A+r}{x-r}, \text{ or } \frac{f_{n-1}x}{x-r}=1+\frac{R_n}{x-r};$$
$$\text{whence } f_{n-1}x=x-r+R_n \ldots\ldots (n).$$

By substituting this value of $f_{n-1}x$ in equation $(n-1)$, we obtain

$$f_{n-2}x=(x-r)^2+R_n(x-r)+R_{n-1}.$$

By substituting this in equation $(n-2)$, we find

$$f_{n-3}x=(x-r)^3+R_n(x-r)^2+R_{n-1}(x-r)+R_{n-2}:$$

and, by similar substitutions, we should at length get

$$fx=(x-r)^n+R_n(x-r)^{n-1}+R_{n-1}(x-r)^{n-2}+\ldots\ldots+R_1 \ldots\ldots (a).$$

Now, if r be the first figure of a root, and x' the rest of the same root, so that $x=r+x'$, and consequently $x-r=x'$, equation (a) becomes $fx=$

$$x'^n+R_nx'^{n-1}+R_{n-1}x'^{n-2}+\ldots\ldots+R_2x'+R_1=0 \ldots\ldots (b).$$

If we now represent the first figure of the value of x' by r', and the rest of it by x'', the value of x'' would be found from the equation,

$$x''^n+R'_nx''^{n-1}+\ldots\ldots+R'_1=0;$$

where R'_n, R'_{n-1}, ..., R'_1, would be found from the first member of (b), in the same manner as R_n, R_{n-1}, &c., were found from fx. This process may be continued as far as we please, and the numbers, r, r', r'', &c., are the first, second, third, &c., figures of the required root.

To find f_1x, f_2x, &c., by a simpler and more convenient form of division, than that which is ordinarily employed, let us assume

$$f_1x=x^{n-1}+p'_1x^{n-2}+p'_2x^{n-3}+\ldots\ldots+p'_{n-1}:$$

then from (1), we have

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_n=$$
$$(x^{n-1}+p'_1x^{n-2}+p'_2x^{n-3}+\ldots\ldots+p'_{n-1})\,(x-r)+\text{R}_1;$$

or, by actual multiplication,

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots\ldots+p_{n-1}x+p_n=$$
$$\begin{array}{r|l|r|l} x^n+p'_1 & x^{n-1}+p'_2 & x^{n-2}+\ldots\ldots+p'_{n-1} & x+\text{R}_1 \\ -r & \quad -rp'_1 & -rp'_{n-2} & \;-rp'_{n-1} \end{array}$$

From this, by equating the coefficients of the like powers of x, and transposing, we get

$$p'_1=p_1+r,\ p'_2=p_2+rp'_1,\ \ldots\ldots p'_{n-1}=p_{n-1}+rp'_{n-2},$$
$$\text{R}_1=p_n+rp'_{n-1}.$$

Hence we have the following form of the division by $x-r$:

$$\begin{array}{ccccc} p_1 & p_2 & p_3\ldots\ldots\ p_{n-1} & p_n(r \\ \underline{r} & \underline{rp'_1} & \underline{rp'_2\ldots\ldots rp'_{n-2}} & \underline{rp'_{n-1}} \\ p'_1 & p'_2 & p'_3\ldots\ldots\ p'_{n-1} & \text{R}_1 \end{array}$$

By operating in a similar manner on all the quantities thus found, except R_1, we should obtain the coefficients of f_2x, and the second remainder, R_2; and by a continuation of the process, all the remainders, R_3, &c., would be found: and these being the coefficients of equation (b), that equation becomes known. Now, the various steps in the operation by Horner's process are nothing else than the steps in the divisions by $x-r$, $x'-r'$, &c., in the manner just shown, the coefficients* alone being employed: and hence, as it would be

* The method of *detached coefficients*, that is, the using of the coefficients alone, without the quantities to which they belong, may be employed, in many instances, with much advantage. Thus, for example, if it were required to multiply $3x^3-5x+4$ by $2x^3+x^2-10$, the work would stand as follows:—

$$\begin{array}{rrrrrrr} 3 & 0 & -5 & 4 & & & \\ \underline{2} & \underline{1} & \underline{0} & \underline{-10} & & & \\ 6 & 0 & -10 & 8 & & & \\ & 3 & 0 & -5 & 4 & & \\ & & & \underline{-30} & \underline{0} & \underline{50} & \underline{-40} \\ 6 & 3 & -10 & -27 & 4 & 50 & -40 \end{array};$$

or $6x^6+3x^5-10x^4-27x^3+4x^2+50x-40$, by supplying the powers of x.

In like manner, to divide $2x^6-5x^5+6x^4-2x^3+5x^2+3$ by x^2-2x+3, we may write the coefficients of the dividend in succession, and after them the coefficients of the divisor, omitting the first as being 1, and changing the signs of the rest, which change converts subtraction into addition throughout the operation. Then the work will stand thus:—

$$\begin{array}{rrrrrrl} 2 & -5 & 6 & -2 & 5 & 0 & 3(2\ -3 \\ & \underline{4} & -6 & 3 & 6 & 9 & -15 \\ & -1 & \underline{-2} & \underline{-4} & \underline{-6} & \underline{10} & -12 \\ & & -2 & -3 & 5 & 19 & \end{array}$$

shown from equation (b), that the value of the figure r' would be estimated by dividing $-R_1$ by R_2, just as the same was shown from equation (3) in the former illustration, the truth of Horner's method is established.

Those who wish to study this subject more particularly, will find Mr. Horner's own papers regarding it in the "Philosophical Transactions" for 1819, and in the new series of Leybourn's "Mathematical Repository," vol. v. They will also find the subject extensively and ably discussed, and much more simply and intelligibly than by Horner himself, in Professor Young's "Treatise on Algebraical Equations."

The following exercises are subjoined for the use of those who may choose to gain practice by working them.

Exer. 1. Required the roots of the equation, $x^2-12x+25=0$. *Answ.* 9·3166248, and 2·6833752.

2. Find the roots of the equation, $x^3+x^2-17x+10=0$. *Answ.* 3·2668179, 0·62567, and $-4·8924879$.

3. Resolve the equation, $x^3-7x-9=0$. *Answ.* One root $=3·1409233$. The other roots are imaginary.

4. What are the roots of the equation, $x^4-4x^3-20x^2+36x-11=0$? *Answ.* 6·236068, 1·763932, $-0·2679492$, and $-3·7320508$.

5. Resolve the equation, $x^4+6x^3-22x^2-198x-243=0$. *Answ.* 5·6055513, and $-1·6055513$. The rest of the roots are imaginary.

6. Find the number which has the sum of its first, second, third, fourth, and fifth powers equal to 100. *Answ.* 2·239643.

Hence, by supplying the proper powers of x after 2, -1, -2, -3, 5, and 19, we have for quotient, $2x^4-x^3-2x^2-3x+5$; and, for remainder, $19x-12$: and, if it were wished, we could easily continue the process in the same manner, so as to obtain any number of terms in x with negative indices. It will be seen, that the division by $x-r$ in Horner's method, proceeds on this principle.

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